

Thermodynamics of dyonic black holes with Thurston horizon geometries

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In five dimensions, we consider a model described by the Einstein gravity with a source given by a scalar field and various Abelian gauge fields with dilatoniclike couplings. For this model, we are able to construct two dyonic black holes whose three-dimensional horizons are modeled by two nontrivial homogeneous Thurston's geometries. The dyonic solutions are of Lifshitz type with an arbitrary value of the dynamical exponent. In fact, the first gauge field ensures the anisotropy asymptotic while the remaining Abelian fields sustain the electric and magnetic charges. Using the Hamiltonian formalism, the mass, the electric, and magnetic charges are explicitly computed. Interestingly enough, the dyonic solutions behave like Chern-Simons vortices in the sense that their electric and magnetic charges turn to be proportional. The extension with an hyperscaling violating factor is also scrutinized where we notice that for specific values of the violating factor, purely magnetic solutions are possible.

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I. INTRODUCTION

During the last decade, some promising efforts have been made to extend the standard adS/CFT correspondence to new areas of physics, and more particularly to physical systems enjoying an anisotropy symmetry. By anisotropy, we mean that the space and the time are allowed to scale with different weights. In this optics, the pioneer works were done in the context of physical models invariant under the Galilean-Schrödinger symmetry [1,2], see also [3] for a geometric approach. Soon after, it was realized that similar holographic considerations can also be translated to the case of scale invariant Lifshitz fixed point systems without Galilean invariance. In this case, the gravity dual metric is commonly known as the Lifshitz spacetime [4] and its representative metric in arbitrary D dimension can be parametrized as

$$ds^2 = -r^{2z} dt^2 + \frac{dr^2}{r^2} + r^2 \sum_{i=1}^{D-2} dx_i^2. \quad (1)$$

In order to avoid as well as possible cumbersome formulas, we have chosen to take the adS radius $l = 1$. It is simple to see that the anisotropic transformations defined by

$$t \rightarrow \lambda^z t, \quad r \rightarrow \frac{1}{\lambda} r, \quad x_i \rightarrow \lambda x_i, \quad (2)$$

are part of the isometry of the Lifshitz metric. Here the constant z which reflects the anisotropy is called the dynamical exponent. In analogy with the adS case, black holes with a Lifshitz asymptotic (1), the so-called *Lifshitz black holes*, would also have a certain interest for holographic considerations. This interest has grew up during the last time as shown by the important literature on the subject, see, e.g., [5–18]. From these different examples it is clear that, in contrast with the adS isotropic case, the Einstein-Hilbert action with eventually a cosmological constant is not enough to sustain the Lifshitz metric. In fact, in standard gravity, Lifshitz black holes can only exist provided the introduction of some extra matter fields [7–9] while higher-order gravity theories with or without matter source may also source the Lifshitz spacetimes, see, e.g., [6,11,12].

Before proceeding, we would like first to enlarge the notion of Lifshitz black holes. In its standard form, the $(D-2)$ -dimensional base manifold of the Lifshitz metric (1) is an Euclidean flat space. This restriction on the manifold ensures that the isometry group of the standard Lifshitz metric (1) contains in addition to the anisotropic transformations (2), the spacetime translations $x_i \rightarrow x_i + c_i$ and $t \rightarrow t + t_0$ as well as the spatial rotation $\vec{x} \rightarrow R\vec{x}$ with $R \in \text{SO}(D-2)$. The algebra of the corresponding generators or equivalently of the Killing vector fields form the so-called Lifshitz algebra. Nevertheless, there also exist black hole solutions with a nonflat base manifold, see, e.g., [19–23], whose asymptotic resembles the Lifshitz one but with a different base manifold

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$$ds^2 = -r^{2z} dt^2 + \frac{dr^2}{r^2} + \sum_{i,j=1}^{D-2} g_{ij}(x, r) dx_i dx_j. \quad (3)$$

For such a metric of course, the isometry group will explicitly depend on the form of the transverse metric g_{ij} . Note that the isometry group of metric-like (3) may not contain the anisotropic dilatations as it occurs for a spherical or hyperboloid transverse metric, [20,24]. There also examples of black hole solutions whose asymptotic forms match with (3) with more than one anisotropic direction [19,23]. In these cases, the standard dilatation transformations (2) are generalized to

$$t \rightarrow \lambda^z t, \quad r \rightarrow \frac{1}{\lambda} r, \quad x_i \rightarrow \lambda^{\alpha_i} x_i, \quad (4)$$

where the coordinates x_i for which $\alpha_i \neq 1$ represent the additional anisotropic directions. We find then appropriate to extend the terminology of Lifshitz black holes to black hole spacetime whose asymptotic metric mimics (3) and which is invariant *at least* under the general dilatation transformations (4). In other words, we only demand that the isometry group contains at least the dilatation generator associated to (4) as well as the generator of time translation. There is a certain interest in extending the notion of Lifshitz black holes as we have done. Indeed, Lifshitz black holes as defined by (3)–(4) have been shown to exit in the case of standard general relativity for dimensions greater than five [19,25]. The restriction on the dimension, namely $D \geq 5$, results from the fact that the horizon's topologies of these generalized Lifshitz solutions are modeled by some of the Thurston's geometries [26] which can only be defined for dimensions greater than three. In fact, as conjectured by Thurston and later on proved by Perelman, any compact orientable three-dimensional Riemannian manifold can be modeled by one of the eight Thurston's geometries¹ which are the Euclidean space \mathbb{E}^3 , the three-sphere \mathbb{S}^3 , the hyperbolic space \mathbb{H}^3 , the products $\mathbb{S}^1 \times \mathbb{S}^2$ and $\mathbb{S}^1 \times \mathbb{H}^2$. In addition, there exist three other possible geometries which are neither of constant curvature nor of product of constant manifolds, called the *Nil* geometry, the *Solv* geometry and the geometry of the universal cover of $\text{SL}_2(\mathbb{R})$. These exotic geometries have the following representative metrics

$$\text{Solv: } d\tilde{s}^2 = x_3^2 dx_1^2 + \frac{1}{x_3^2} dx_2^2 + \frac{1}{x_3^2} dx_3^2, \quad (5a)$$

$$\text{Nil: } d\tilde{s}^2 = dx_1^2 + dx_2^2 + (dx_3 - x_1 dx_2)^2, \quad (5b)$$

$$\text{SL}_2(\mathbb{R}): d\tilde{s}^2 = \frac{1}{x_1^2} (dx_1^2 + dx_2^2) + \left(dx_3 + \frac{dx_2}{x_1} \right)^2, \quad (5c)$$

¹In fact, these eight three-dimensional Thurston's geometries can be extended in dimensions $D > 3$.

which can be schematically written as

$$d\tilde{s}^2 = \sum_{I=1}^3 \omega_I^2, \quad (6)$$

where the ω_I are the corresponding left-invariant one-forms with $I = \{1, 2, 3\}$. For latter convenience, in what follows, we will use the notation $I = (i, 3)$ where i ranges from 1 to 2.

In order to be self-contained, we report the generalized Lifshitz black holes solutions of standard five-dimensional general relativity, i.e., $G_{\mu\nu} + \Lambda g_{\mu\nu} = 0$ found in [19] and having horizon's topologies described by the three-dimensional Solv and Nil geometries. In fact, these solutions can be represented as follows

$$ds^2 = -r^{2z} f(r) dt^2 + \frac{dr^2}{r^2 f(r)} + \sum_{I=1}^3 a_I r^{2q_I} \omega_I^2, \quad (7)$$

where the a_I 's are constants that allow the introduction of an eventual additional scale. In the case of the Solv black hole solution, the set of parameters reads

$$\begin{aligned} \text{Solv: } f(r) &= 1 - \frac{M}{r^3}, \\ \left\{ z &= 1, q_i = 1, q_3 = 0, a_i = 1, a_3 = \frac{2}{3} \right\}, \end{aligned} \quad (8)$$

while for black hole solution with Nil's horizon topology, the parameters are given by

$$\begin{aligned} \text{Nil: } f(r) &= 1 - \frac{M}{r^{11/2}}, \\ \left\{ z &= 3/2, q_i = 1, q_3 = 2, a_i = 1, a_3 = \frac{11}{2} \right\}. \end{aligned} \quad (9)$$

It is clear that both solutions satisfy asymptotically the requirements given in (3)–(4). More precisely, the Solv's solution (7)–(8) is asymptotically invariant under a one-parametric Lifshitz generalized transformations (4) defined by

$$\begin{aligned} t &\rightarrow \lambda t, & r &\rightarrow \frac{1}{\lambda} r, & x_1 &\rightarrow \lambda^{1-\alpha} x_1, \\ x_2 &\rightarrow \lambda^{1+\alpha} x_2, & x_3 &\rightarrow \lambda^\alpha x_3, \end{aligned} \quad (10)$$

while for the Nil's solution (7)–(9), one has two anisotropic directions

$$t \rightarrow \lambda^{\frac{3}{2}} t, \quad r \rightarrow \frac{1}{\lambda} r, \quad x_i \rightarrow \lambda x_i, \quad x_3 \rightarrow \lambda^2 x_3. \quad (11)$$

In the present work, we propose to find the dyonic version of the Solv (7)–(8) and of the Nil's solution (7)–(9). The interests for such study are multiple. First of all,

charged Lifshitz black holes are known to have rather unconventional thermodynamical properties whose range is largely spread from solutions with a Reissner-Nordstrom-like behavior [24] to zero-mass charged solutions [7] including extremal solutions [27]. The richness of these properties is essentially due to the difficulty of “charging” the known Lifshitz solutions. This is in contrast with the adS situation where an important class of charged adS black hole solutions arise simply from the neutral configurations turning on the Maxwell action. The situation is radically different for the Lifshitz black holes where all the known electrically charged Lifshitz black holes solutions of Einstein gravity require, in addition to the Maxwell potential, some extra fields materialized by scalar field with a dilatonic coupling [24] or a massive Proca field [7,17] or by considering nonminimal coupling [28]. In other words, the Maxwell field alone is incompatible with the Lifshitz asymptotic for the Einstein-Maxwell model. Nevertheless, this problem can be circumvented in higher dimensions $D \geq 4$ where quadratic corrections of the Einstein gravity can accommodate Maxwell charged Lifshitz black holes [12]. The lesson learned from these examples is that the presence of extra parameters in the action permits to soften the incompatibility between the Maxwell potential and the Lifshitz asymptotic. We would like to explore the relevance of this observation in order to achieve our task of charging the Solv and the Nil’s solutions. More specifically, we will consider a model described by the Einstein gravity with a negative cosmological constant together with a scalar field and various (at least 3) $U(1)$ gauge fields with dilatoniclike couplings. Indeed, dilatonic sources are usually good laboratories for investigating charged black holes, see, e.g., [29–31]. As shown below, the presence of more than one $U(1)$ gauge field is mandatory in order to ensure the Lifshitz asymptotic as well as the presence of the electric and magnetic charges. In fact, the first gauge field guarantees the Lifshitz asymptotic while the remaining Abelian fields sustain the electric and magnetic charges.² We will also see that the dyonic extensions of the of the Solv (7)–(8) and of the Nil’s solution (7)–(9) with a multidilatonic source present some interesting features. For example, the introduction of the dilatonic source will extend the range of the dynamical exponent. Indeed, while the vacuum Solv’s (resp. Nil’s) dyonic solution requires $z = 1$ (resp. $z = \frac{3}{2}$), their dyonic extensions will exist for a Lifshitz dynamical exponent $z \geq 1$ (resp. $z \geq \frac{3}{2}$). Also, the dyonic solutions presented below are quite different from those existing in the current literature in the sense that their electric and magnetic charges are proportional. This in turn implies that there does not exist a purely electric or magnetic limit as it is the

²Note that for the purely electrically Lifshitz charged black holes with planar, spherical, or hyperboloid horizon topology [24], two dilatonic fields were at least required.

case for the four-dimensional dyonic Reissner-Nordstrom solution.

The plan of the paper is organized as follows. In the next section, we will explicitly present the five-dimensional model, its field equations as well as the ansatz we will consider. In Secs. III and IV, we will display the dyonic extensions of the Solv (7)–(8) and of the Nil’s solution (7)–(9). A detailed analysis of their thermodynamic features will be provided showing that their electric and magnetic charges are in fact proportional. In each case, we will check that the electromagnetic version of the first law of thermodynamics is satisfied. In Sec. V, we will extend these results to the so-called hyperscaling violating case with Solv and Nil’s horizon topologies. In this case, the Lagrangian model involves a Liouville potential but without the cosmological constant. Interestingly enough, for precise values of the hyperscaling violation factor the electric contribution can be canceled yielding to purely magnetically charged configurations. Finally, the last section is dedicated to our conclusions.

II. ACTION, FIELD EQUATIONS, AND ANSATZ WITH THURSTON GEOMETRIES

As anticipated in the introduction, the five-dimensional action we consider is given by the standard Einstein-Hilbert action with a cosmological constant together with N $U(1)$ gauge fields with dilatoniclike couplings,

$$S = \int d^5x \sqrt{-g} \left[\frac{R - 2\Lambda}{2} - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{4} \sum_{i=1}^N e^{\lambda_i \phi} F_{(i)\mu\nu} F_{(i)}^{\mu\nu} \right], \quad (12)$$

where as shown below $N = 3$ in the case of the Solv’s solution and $N = 4$ for the dyonic Nil’s solution.

The equations of motions obtained by varying the action with respect to the metric, the gauge vector fields, and the scalar field respectively read

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = T_{\mu\nu}, \quad (13a)$$

$$\nabla_\mu (e^{\lambda_i \phi} F_{(i)}^{\mu\nu}) = 0, \quad (13b)$$

$$\square \phi = \sum_{i=1}^N \left(\frac{\lambda_i}{4} e^{\lambda_i \phi} F_{(i)\sigma\rho} F_{(i)}^{\sigma\rho} \right), \quad (13c)$$

where the energy-momentum tensor $T_{\mu\nu}$ is defined as

$$T_{\mu\nu} = \left(\nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} g_{\mu\nu} \nabla_\sigma \phi \nabla^\sigma \phi \right) + \sum_{i=1}^N \left(e^{\lambda_i \phi} F_{(i)\mu\sigma} F_{(i)\nu}^{\sigma} - \frac{1}{4} g_{\mu\nu} e^{\lambda_i \phi} F_{(i)\sigma\rho} F_{(i)}^{\sigma\rho} \right). \quad (14)$$

In this paper, we will consider an ansatz for the metric of the form (7), and in order for the metric ansatz (7) to be asymptotically Lifshitz in the sense of (3)–(4), we will require that $\lim_{r \rightarrow \infty} f(r) = 1$ and the left-invariant one-forms ω_I scale homogeneously as $\omega_I \rightarrow \lambda^{q_I} \omega_I$ under the dilatation transformations $t \rightarrow \lambda^z t$ and $r \rightarrow \frac{1}{\lambda} r$.

III. ELECTROMAGNETIC CHARGED SOLUTION WITH A SOLV'S HORIZON TOPOLOGY

We first report a charged dyonic black hole solution of the field equations (13) for which the line element has a Solv's horizon topology parameterized as follows

$$\begin{aligned}
 ds^2 = & -r^{2z} f(r) dt^2 + \frac{dr^2}{r^2 f(r)} + r^2 x_3^2 dx_1^2 \\
 & + r^2 \frac{dx_2^2}{x_3^2} + \left(\frac{2}{z+2} \right) \frac{dx_3^2}{x_3^2}, \\
 f(r) = & 1 - m \left(\frac{r_h}{r} \right)^{z+2} + (m-1) \left(\frac{r_h}{r} \right)^{2z+2}. \quad (15)
 \end{aligned}$$

The matter fields associated to this spacetime metric read

$$\begin{aligned}
 e^\phi = & r^{\sqrt{2(z-1)}}, \quad F_{(1)rt} = \sqrt{(z+2)(z-1)} r^{z+1}, \\
 F_{(2)rt} = & \sqrt{z(m-1)} \left(\frac{r_h}{r} \right)^{z+1}, \\
 F_{(3)x_1 x_2} = & \sqrt{z(m-1)} r_h^{z+1}, \quad (16)
 \end{aligned}$$

and the solution exists provided that the coupling constants are chosen as

$$\begin{aligned}
 \Lambda = & -\frac{(z+2)^2}{2}, \quad \lambda_1 = -\frac{4}{\sqrt{2(z-1)}}, \\
 \lambda_2 = & \sqrt{2(z-1)}, \quad \lambda_3 = -\sqrt{2(z-1)}. \quad (17)
 \end{aligned}$$

Before providing a complete thermodynamics analysis of the Solv's dyonic solution, some additional comments are needed. First, the existence of the Solv's solution is ensured for at least three Abelian gauge fields with dilatoniclike couplings. Second, we find judicious to parametrize the metric solution as in (15) which makes clear that r_h stands for the location of the event horizon. Nevertheless, as shown below, the two integration constants m and r_h will be identified with the mass, the electric and magnetic charges. In addition, the metric function appearing in (15) will have a Reissner-Nordstrom-like form. It will also become clear after the thermodynamics analysis that the electric and magnetic charges are proportional which in turn explains the mismatch between the number of integration constants and of charges. The range of the Lifshitz dynamical exponent is given by $z \geq 1$. In fact, even if the coupling constant λ_1 as defined in (17) blows up in the limit $z = 1$, the first dilaton Lagrangian in the action $e^{\lambda_1 \phi} F_{(1)\mu\nu} F^{(1)\mu\nu} \rightarrow 0$

as $z \rightarrow 1$. More precisely, the limiting adS case $z = 1$ reduces to the dyonic solution recently found in [32] in the absence of the scalar field or to the vacuum solution for $m = 1$ (7)–(8), see Ref. [19]. Hence, interestingly enough, the fact of turning on the dilatonic source permits to extend the range of the dynamical exponent to be $z \geq 1$. Consequently, the asymptotic metric is invariant under the following one-parametric Lifshitz generalized dilatation transformations (4) extending those of the vacuum sector (10) and defined by

$$\begin{aligned}
 t \rightarrow \lambda^z t, \quad r \rightarrow \frac{1}{\lambda} r, \quad x_1 \rightarrow \lambda^{1-\alpha} x_1, \\
 x_2 \rightarrow \lambda^{1+\alpha} x_2, \quad x_3 \rightarrow \lambda^\alpha x_3.
 \end{aligned}$$

Also, the scalar field is defined up to a constant c , and this constant can be put to zero without any loss of generality since it is a symmetry of the dilaton action represented as $\phi \rightarrow \phi - c$ and $A_{(i)} \rightarrow e^{\frac{\lambda_i c}{2}} A_{(i)}$. Finally, we would like to point out an interesting fact concerning the electro-magnetic duality. In the hyperplane defined by $x_3 = \text{cst}$, the electric and magnetic fields are dual in the sense that

$$\star F_{(2)} = F_{(3)}, \quad (18)$$

where the Hodge dual operator \star is defined for the four-dimensional metric defined by $x_3 = \text{cst}$.

We now turn to the thermodynamics study of the Solv's charged dyonic solution (15)–(17). As was shown in [33], the partition function for a thermodynamics ensemble may be identified with the Euclidean path integral in the saddle point approximation around the Euclidean continuation of the solution. In the present case, we will deal with a reduced action principle with a static Euclidean metric endowed by a Solv's horizon topology. More precisely, the Euclidean ansatz for this mini superspace configuration is given by the following line element

$$\begin{aligned}
 ds^2 = & N^2(r) F(r) d\tau^2 + \frac{dr^2}{F(r)} + r^2 x_3^2 dx_1^2 + r^2 \frac{dx_2^2}{x_3^2} \\
 & + \left(\frac{2}{z+2} \right) \frac{dx_3^2}{x_3^2}, \quad (19)
 \end{aligned}$$

with matter fields given as

$$\begin{aligned}
 A_{(i)\mu} dx^\mu = & A_{(i)\tau}(r) d\tau, \\
 A_{(3)\mu} dx^\mu = & A_{(3)x_1}(x_2) dx_1 + A_{(3)x_2}(x_1) dx_2, \quad \phi = \phi(r),
 \end{aligned}$$

with $i = 1, 2$. In the Euclidean continuation, the range of the radial coordinate is from the horizon r_h to infinity and the Euclidean time $\tau = it$ is compactified as $\tau \in [0, \beta]$ where β stands for the inverse of the temperature $\beta = T^{-1}$. As usual, using this ansatz, a reduced action can be written

in a ‘‘Hamilton form.’’ Nevertheless, because of the presence of the magnetic field $F_{(3)}$, we will carefully derive this Hamilton form in various steps.

The Euclidean action denoted I_E for the previous ansatz is schematically decomposed in five pieces as

$$I_E = I_{\text{EH}} + I_{\text{kin}} + \sum_{i=1}^2 I_{F_{(i)}} + I_{F_{(3)}} + B_E. \quad (20)$$

The first two terms correspond to the Einstein-Hilbert piece and the kinetic term of the scalar field, the $I_{F_{(i)}}$'s are the electric dilatonic parts of the action while $I_{F_{(3)}}$ stands for the magnetic part(s) and B_E is a boundary term fixed in such a way that the reduced action I_E has a well-defined extremum,

that is $\delta I_E = 0$. As shown below, the Euclidean action on-shell reduces to the boundary term and is related to the Gibbs free energy \mathcal{G} as

$$I_E = \beta \mathcal{G} = \beta(\mathcal{M} - \Phi_e \mathcal{Q}_e - \Phi_m \mathcal{Q}_m) - \mathcal{S}, \quad (21)$$

where \mathcal{M} is the mass, \mathcal{S} the entropy, Φ_e (resp. Φ_m) corresponds to the electric (resp. magnetic) potential and \mathcal{Q}_e (resp. \mathcal{Q}_m) represents the electric (resp. magnetic) charge. Note that we opt for the formalism of the grand canonical ensemble where the temperature as well the electric and magnetic potentials are fixed.

For the dyonic Solv's solution, we found that the different pieces of the reduced Euclidean action are given by

$$\begin{aligned} I_{\text{EH}} &= |\Omega_{\text{Solv}}| \frac{\beta}{2} \sqrt{\frac{2}{z+2}} \int_{r_h}^{\infty} N(r) [2\Lambda r^2 + r^2 z + 2rF(r)' + 2r^2 + 2F(r)] dr, \\ I_{\text{kin}} &= |\Omega_{\text{Solv}}| \frac{\beta}{2} \sqrt{\frac{2}{z+2}} \int_{r_h}^{\infty} N(r) [r^2 F(r) (\phi(r)')^2] dr, \\ I_{F_{(i)}} &= \beta \int \left[A_{(i)\tau} \partial_r \mathcal{P}_{(i)}(r, x_i, x_3) + \frac{N(r)}{2r^2} x_3 \sqrt{\frac{z+2}{2}} e^{-\lambda_i \phi} \mathcal{P}_{(i)}(r, x_i, x_3)^2 \right] dx_1 dx_2 dx_3 dr, \\ I_{F_{(3)}} &= \frac{\beta}{2} \sqrt{\frac{2}{z+2}} \int \frac{N(r)}{r^2 x_3} e^{\lambda_3 \phi} (\partial_{x_1} A_{(3)x_2} - \partial_{x_2} A_{(3)x_1})^2 dx_1 dx_2 dx_3 dr. \end{aligned}$$

In these expressions, we have defined $|\Omega_{\text{Solv}}|$ to be the volume element of the compact Solv's spacetime (5a), that is

$$|\Omega_{\text{Solv}}| = \int_{\Omega_1 \times \Omega_2 \times \Omega_3} dx_1 dx_2 \frac{dx_3}{x_3} = |\Omega_1| |\Omega_2| \ln |\Omega_3|, \quad (22)$$

where the Ω_I 's for $I = 1, 2, 3$ stand for the compact ranges of the horizon coordinates x_I . On the other hand, the $\mathcal{P}_{(i)}$'s are the conjugate momenta of the electric potential fields $A_{(i)}$ for $i = 1, 2$. We can note that the last two integrals $I_{F_{(i)}}$ and $I_{F_{(3)}}$ still involve the four-dimensional volume element; this is due to the fact that the electric conjugate momentum and the magnetic gauge field are allowed to depend on the horizon coordinates x_i and x_3 . Nevertheless, this dependence can be specified through the field equations associated to the Euclidean action. Indeed, the variation of I_E with respect to the conjugate momenta $\mathcal{P}_{(i)}$ implies that the conjugate momenta are separable in the following way

$$\begin{aligned} \mathcal{P}_{(i)} &= \frac{\bar{\mathcal{P}}_{(i)}(r)}{x_3} \quad \text{with} \\ \bar{\mathcal{P}}_{(i)}(r) &= r^2 \sqrt{\frac{2}{z+2}} \frac{e^{\lambda_i \phi}}{N} \partial_r A_{(i)\tau}, \quad \text{for } i = 1, 2. \end{aligned} \quad (23)$$

On the other hand, the variation with respect to the magnetic gauge field $A_{(3)}$ forces the magnetic gauge field to be linear in x_i . Hence, under these last considerations, the reduced action I_E can be written in a Hamilton form as

$$\begin{aligned} I_E &= \beta |\Omega_{\text{Solv}}| \int_{r_h}^{\infty} \left(N \mathcal{H} + \sum_{i=1}^2 A_{(i)\tau} \bar{\mathcal{P}}_{(i)} \right) dr + B_E, \\ \mathcal{H} &= \sqrt{\frac{2}{z+2}} \left[\Lambda r^2 + r^2 \left(1 + \frac{z}{2} \right) + F + rF' + \frac{r^2}{2} F(\phi')^2 \right. \\ &\quad \left. + \sum_{i=1}^2 \left(\frac{z+2}{4r^2} \right) e^{-\lambda_i \phi} \bar{\mathcal{P}}_{(i)}^2 + \frac{e^{\lambda_3 \phi}}{2r^2} (F_{(3)x_1 x_2})^2 \right]. \end{aligned} \quad (24)$$

It is reassuring to check that the field equations obtained by varying the reduced action I_E (24) with respect to $N, F, \phi, \bar{\mathcal{P}}_{(i)}, A_{(i)}$ and $A_{(3)}$ are consistent with the original equations of motion (13).

Now, we are in position to determine the boundary term B_E that will encode all the thermodynamics features of the solution. This term is fixed by requiring that the total action has an extremum $\delta I_E = 0$ with

$$\delta I_E = \beta |\Omega_{\text{Solv}}| \left[\sqrt{\frac{2}{z+2}} N(r) \delta F + r^2 F \phi' \delta \phi + \sum_{i=1}^2 A_{(i)\tau} \delta \bar{\mathcal{P}}_{(i)} \right]_{r_h}^{\infty} + \delta I_{F_{(3)}} + \delta B_E.$$

The variation of the magnetic part $\delta I_{F_{(3)}}$ must be done with care,

$$\begin{aligned} \delta I_{F_{(3)}} &= \beta \sqrt{\frac{2}{z+2}} \ln |\Omega_3| \int dr \frac{N e^{\lambda_3 \phi}}{r^2} \epsilon^{ij} [\partial_{x_i} A_{(3)x_j} \delta A_{(3)x_j} | \Omega_j]_{x_i \in \Omega_i} \\ &= \beta |\Omega_{\text{Solv}}| \sqrt{\frac{2}{z+2}} \sqrt{z(m-1)} r_h^{z+1} \delta(\sqrt{z(m-1)} r_h^{z+1}) \int_{r_h}^{\infty} dr \frac{N e^{\lambda_3 \phi}}{r^2}, \end{aligned}$$

where ϵ^{ij} is the totally antisymmetric tensor with $\epsilon^{12} = 1$. Note that in the second line, we have used the fact that $A_{(3)}$ is linear in x_i (16), and this also explains the reason for which the volume element of the Solv's geometry (22) appears. This variation of the magnetic piece is analogous to what occurs in the magnetically charged Reissner-Nordstrom solution or to what have been done recently in the case of adS_4 dyonic black holes [34]. For the Solv's solution (15)–(17) with metric functions N and F identified as

$$\begin{aligned} N(r) &= r^{z-1}, \\ F(r) &= r^2 \left[1 - m \left(\frac{r_h}{r} \right)^{z+2} + (m-1) \left(\frac{r_h}{r} \right)^{2z+2} \right], \end{aligned}$$

a straightforward computation permits to obtain the variation of the boundary term

$$\begin{aligned} \delta B_E &= \beta |\Omega_{\text{Solv}}| \left[\sqrt{\frac{2}{z+2}} \left(\delta(m r_h^{z+2}) - \frac{2\pi}{\beta} \delta r_h^2 \right) \right. \\ &\quad - [A_{(2)\tau}(\infty) - A_{(2)\tau}(r_h)] \delta \left(\sqrt{\frac{2z(m-1)}{z+2}} r_h^{z+1} \right) \\ &\quad \left. - \sqrt{\frac{2(m-1)}{z+2}} r_h \delta(\sqrt{m-1} r_h^{z+1}) \right], \end{aligned}$$

where, as usual, in order to avoid conical singularity, we require that $N \delta F|_{r_h} = -\frac{4\pi}{\beta} \delta r_h$. We also identify the electric potential Φ_e as the difference of the gauge field between the infinity and the event radius, i.e.,

$$\Phi_e = A_{(2)\tau}(\infty) - A_{(2)\tau}(r_h) = \sqrt{\frac{m-1}{z}} r_h,$$

and hence the second piece in δB_E is identified with the electric variation $-\Phi_e \delta \mathcal{Q}_e$. Analogously, the last variation in δB_E must correspond to the magnetic variation $-\Phi_m \delta \mathcal{Q}_m$, see (21). Nevertheless, contrary to the electric part, there is *a priori* no way of identifying the magnetic potential, and hence the magnetic potential and charge can only be determined up to two constants, namely

$$\begin{aligned} \Phi_m &= A \sqrt{m-1} r_h, & \mathcal{Q}_m &= B \sqrt{m-1} |\Omega_{\text{Solv}}| r_h^{z+1}, \\ AB &= \sqrt{\frac{2}{z+1}}. \end{aligned}$$

We can note from now that independently of the fact that the constants A and B are not fixed, it is clear that the electric and magnetic charge are proportional (see below for the electric charge). However, since the electric and magnetic variations are equal $-\Phi_e \delta \mathcal{Q}_e = -\Phi_m \delta \mathcal{Q}_m$, and because of the electromagnetic duality in the hyperplane defined by $x_3 = \text{cst}$ (18), one can suppose that $\Phi_e = \Phi_m$ and $\mathcal{Q}_e = \mathcal{Q}_m$. Finally, in the formalism of the grand canonical ensemble the boundary term can be expressed as

$$\begin{aligned} B_E &= \beta |\Omega_{\text{Solv}}| \left[\sqrt{\frac{2}{z+2}} m r_h^{z+1} \right. \\ &\quad \left. - (\Phi_e + \Phi_m) \left(\sqrt{\frac{2z(m-1)}{z+2}} r_h^{z+1} \right) \right] \\ &\quad - 2\pi \sqrt{\frac{2}{z+2}} |\Omega_{\text{Solv}}| r_h^2. \end{aligned}$$

Since the on-shell Euclidean action reduces to the boundary term $I_{E|_{\text{on-shell}}} = B_E$, the different thermodynamics quantities can easily be determined through (21) yielding to

$$\begin{aligned} \mathcal{M} &= |\Omega_{\text{Solv}}| \sqrt{\frac{2}{z+2}} m r_h^{z+2}, & T &= \frac{(2z+2-mz)r_h^z}{4\pi}, \\ \mathcal{S} &= 2\pi |\Omega_{\text{Solv}}| \sqrt{\frac{2}{z+2}} r_h^2, & \Phi_e = \Phi_m &= \sqrt{\frac{m-1}{z}} r_h, \\ \mathcal{Q}_e = \mathcal{Q}_m &= |\Omega_{\text{Solv}}| \sqrt{\frac{2z(m-1)}{z+2}} r_h^{z+1}. \end{aligned} \quad (25)$$

It is straightforward to check the validity of the first law of thermodynamics

$$d\mathcal{M} = T d\mathcal{S} + \Phi_e d\mathcal{Q}_e + \Phi_m d\mathcal{Q}_m. \quad (26)$$

Just to conclude this section, we note that the metric solution (15) can now be rewritten in the Reissner-Nordstrom-like form as

$$\begin{aligned}
ds^2 &= -r^{2z}f(r)dt^2 + \frac{dr^2}{r^2f(r)} + r^2x_3^2dx_1^2 \\
&\quad + r^2\frac{dx_2^2}{x_3^2} + \left(\frac{2}{z+2}\right)\frac{dx_3^2}{x_3^2}, \\
f(r) &= 1 - \sqrt{\frac{z+2}{2}} \frac{\mathcal{M}}{|\Omega_{\text{Solv}}|r^{z+2}} + \frac{(z+2)}{4z|\Omega_{\text{Solv}}|^2} \frac{\mathcal{Q}_e^2 + \mathcal{Q}_m^2}{r^{2(z+1)}}. \quad (27)
\end{aligned}$$

From this last expression, one observes that in the adS limit $z = 1$, even if the fall off of the mass term is more faster than in the standard five-dimensional Reissner-Nordstrom case, one still gets a finite and nonzero value of the mass for the Solv's dyonic solution.

IV. PURELY LIFSHITZ DYONIC SOLUTION WITH A NIL'S HORIZON TOPOLOGY

We now present the dyonic extension of the Nil's solution (7)–(9). In this case, it is possible to find the following class of solution with a line element that reads

$$\begin{aligned}
ds^2 &= -r^{2z}f(r)dt^2 + \frac{dr^2}{r^2f(r)} + r^2dx_1^2 + r^2dx_2^2 \\
&\quad + (z+4)r^4(dx_3 - x_1dx_2)^2, \\
f(r) &= 1 - m\left(\frac{r_h}{r}\right)^{z+4} + (m-1)\left(\frac{r_h}{r}\right)^{2z+4}. \quad (28)
\end{aligned}$$

The gauge and scalar fields associated to this line element are given by

$$\begin{aligned}
e^\phi &= r\sqrt{2(2z-3)}, \quad F_{(1)rt} = \frac{\sqrt{2(2z-3)(z+4)}}{2} r^{z+3}, \\
F_{(2)rt} &= \frac{\sqrt{2z(m-1)}r_h^{z+2}}{r^{z+1}}, \\
F_{(3)x_1x_2} &= -x_1\sqrt{z(m-1)(z+4)}r_h^{z+2}, \\
F_{(3)x_1x_3} &= \sqrt{z(m-1)(z+4)}r_h^{z+2}, \\
F_{(4)x_2x_3} &= \sqrt{z(m-1)(z+4)}r_h^{z+2}, \quad (29)
\end{aligned}$$

and the parameters must be chosen as

$$\begin{aligned}
\Lambda &= -\frac{(z+4)(z+3)}{2}, \quad \lambda_1 = -\frac{8}{\sqrt{2(2z-3)}}, \\
\lambda_2 &= \frac{2(z-2)}{\sqrt{2(2z-3)}}, \quad \lambda_3 = \lambda_4 = -\frac{2(z-1)}{\sqrt{2(2z-3)}}.
\end{aligned}$$

Few comments can be made concerning this dyonic solution with Nil's horizon. First, the Nil dyonic solution requires at least four $U(1)$ gauge fields with dilatonic couplings and is valid for a dynamical exponent $z \geq \frac{3}{2}$. As before, the limiting case $z = \frac{3}{2}$ with $m = 1$ (that is without electromagnetic charges) reduces to the vacuum Nil's

solution (7)–(9). Second, the solution does not exhibit a such electromagnetic duality (18) as was for the Solv's solution. In addition, the dyonic Nil's solution can be likened to a Lifshitz black hole whose asymptotic symmetries contain a generalized dilatation transformation with two anisotropic direction (4) given by

$$t \rightarrow \lambda^z t, \quad r \rightarrow \frac{1}{\lambda} r, \quad x_i \rightarrow \lambda x_i, \quad x_3 \rightarrow \lambda^2 x_3.$$

Finally, as in the previous case, the two integration constants will be shown to represent the mass and the electric/magnetic charges.

Let us now study the thermodynamics properties of the Nil's solution. Following the same lines as those presented in details for the Solv's solution, we consider the following Euclidean ansatz

$$\begin{aligned}
ds^2 &= N^2(r)F(r)d\tau^2 + \frac{dr^2}{F(r)} + r^2dx_1^2 + r^2dx_2^2 \\
&\quad + (z+4)r^4(dx_3 - x_1dx_2)^2, \\
A_{(i)\mu}dx^\mu &= A_{(i)\tau}(r)d\tau, \quad A_{(3)\mu}dx^\mu = A_{(3)I}(x_J)dx^I, \\
A_{(4)\mu}dx^\mu &= A_{(4)I}(x_J)dx^I, \quad \phi = \phi(r).
\end{aligned}$$

For this class of ansatz, the Euclidean action (20) is decomposed as

$$\begin{aligned}
I_{\text{EH}} &= |\Omega_{\text{Nil}}|\beta\sqrt{z+4} \\
&\quad \times \int_{r_h}^{\infty} N(r) \left[7r^2F + r^4\left(\frac{z}{4} + \Lambda + 1\right) + 2r^3F' \right] dr, \\
I_{\text{kin}} &= |\Omega_{\text{Nil}}|\frac{\beta}{2}\sqrt{\frac{2}{z+2}} \int_{r_h}^{\infty} N(r)[r^4F(r)(\phi(r)')^2] dr, \\
I_{F_{(i)}} &= |\Omega_{\text{Nil}}|\beta \int \left[A_{(i)\tau}\partial_r\bar{\mathcal{P}}_{(i)} + \frac{N}{r^4\sqrt{z+4}} e^{-\lambda_i\phi}\bar{\mathcal{P}}_{(i)}^2 \right] dr,
\end{aligned}$$

and the magnetic pieces read

$$\begin{aligned}
I_{F_{(3,4)}} &= \frac{\beta}{2}\sqrt{z+4} \int \left\{ Ne^{\lambda_3\phi} \left[F_{(3)x_1x_2}^2 + 2x_1F_{(3)x_1x_2}F_{(3)x_1x_3} \right. \right. \\
&\quad \left. \left. + \left(\frac{1}{r^2(z+4)} + x_1^2\right)F_{(3)x_1x_3}^2 \right] \right. \\
&\quad \left. + Ne^{\lambda_4\phi} \frac{F_{(4)x_2x_3}^2}{r^2(z+4)} \right\} dx_1dx_2dx_3dr.
\end{aligned}$$

In these expressions, we have defined $|\Omega_{\text{Nil}}|$ to be the volume element of the Nil geometry (5)

$$|\Omega_{\text{Nil}}| = \int_{\Omega_1 \times \Omega_2 \times \Omega_3} dx_1dx_2dx_3 = |\Omega_1||\Omega_2||\Omega_3|. \quad (30)$$

The dependence of the magnetic field strengths $F_{(3)}$ and $F_{(4)}$ on the Thurston's coordinates x_I can be fixed by

varying the total action (20) with respect to $A_{(3)I}$ and $A_{(4)I}$. In doing so, one obtains that

$$F_{(3)x_1x_3} = -\frac{1}{x_1}F_{(3)x_1x_2} = \text{cst}, \quad F_{(4)x_2x_3} = \text{cst}.$$

Using this last result, the x_I -dependence of the magnetic action $I_{F_{(3,4)}}$ is canceled out, and hence the reduced Euclidean action (20) can be written in Hamilton form depending only on the radial coordinate as

$$I_E = \beta|\Omega_{\text{Nil}}| \int_{r_h}^{\infty} \left(N\mathcal{H} + \sum_{i=1}^2 A_{(i)\tau} \bar{\mathcal{P}}'_{(i)} \right) dr + B_E,$$

where the Hamiltonian reads

$$\begin{aligned} \mathcal{H} = & \sqrt{z+4} \left[7r^2 F + r^4 \left(\frac{z}{4} + \Lambda + 1 \right) + 2r^3 F' + \frac{r^4}{2} F(\phi')^2 \right. \\ & + \sum_{i=1}^2 \frac{e^{-\lambda_i \phi}}{(z+4)r^4} \bar{\mathcal{P}}_{(i)}^2 + \frac{e^{\lambda_3 \phi}}{2r^2(z+4)} F_{(3)x_1x_3}^2 \\ & \left. + \frac{e^{\lambda_4 \phi}}{2r^2(z+4)} F_{(4)x_2x_3}^2 \right]. \end{aligned}$$

As explained before, the boundary term encodes all the thermodynamical features of the solution. The boundary is fixed by requiring that the on-shell Euclidean action has a well-defined extremum $\delta I_E = 0$ with

$$\begin{aligned} \delta I_E = & \beta|\Omega_{\text{Nil}}| \left[\sqrt{z+4} N (2r^3 \delta F + r^4 F \phi' \delta \phi) \right. \\ & \left. + \sum_{i=1}^2 A_{(i)\tau} \delta \bar{\mathcal{P}}_{(i)} \right]_{r_h}^{\infty} + \delta I_{F_{(3,4)}} + \delta B_E, \end{aligned}$$

where $\delta I_{F_{(3,4)}}$ stands for the variation of the magnetic dilaton parts of the action

$$\begin{aligned} \delta I_{F_{(3,4)}} = & \frac{\beta}{\sqrt{z+4}} \int dr \frac{N e^{\lambda_3 \phi}}{r^2} \sigma_{(3)}^{ij} [\partial_i A_{(3)j} \delta A_{(3)j} | \Omega_2 | | \Omega_j |]_{x_i \in \Omega_i} \\ & + \frac{\beta}{\sqrt{z+4}} \int dr \frac{N e^{\lambda_4 \phi}}{r^2} \sigma_{(4)}^{ij} [\partial_i A_{(4)j} \delta A_{(4)j} | \Omega_1 | | \Omega_j |]_{x_i \in \Omega_i}, \end{aligned}$$

where the nonvanishing components of $\sigma_{(3)}^{ij}$ and $\sigma_{(4)}^{ij}$ are given by $\sigma_{(3)}^{13} = -\sigma_{(3)}^{31} = 1$ and $\sigma_{(4)}^{23} = -\sigma_{(4)}^{32} = 1$.

After some computations, for the Nil's solution with metric functions and conjugate momenta given by

$$N(r) = r^{z-1},$$

$$F(r) = r^2 \left[1 - m \left(\frac{r_h}{r} \right)^{z+4} + (m-1) \left(\frac{r_h}{r} \right)^{2z+4} \right],$$

$$\bar{\mathcal{P}}_{(i)} = \frac{\sqrt{z+4} r^4}{N(r)} e^{\lambda_i \phi} \partial_r A_{(i)\tau},$$

one obtains for the variation of the boundary term

$$\begin{aligned} \delta B_E = & \beta|\Omega_{\text{Nil}}| \left[\sqrt{z+4} \left(\delta(2mr_h^{z+4}) - \frac{2\pi}{\beta} \delta r_h^4 \right) \right. \\ & - \Phi_e \delta(\sqrt{2z(z+4)(m-1)} r_h^{z+2}) \\ & \left. - 2\sqrt{(m-1)} r_h^2 \delta(\sqrt{(m-1)(z+4)} r_h^{z+2}) \right], \end{aligned}$$

where the electric potential is given by $\Phi_e = \sqrt{\frac{2(m-1)}{z}} r_h^2$. Finally, the boundary term in the formalism of the grand canonical ensemble is expressed as

$$\begin{aligned} B_E = & \beta|\Omega_{\text{Nil}}| [2\sqrt{z+4} m r_h^{z+4} - \Phi_e \mathcal{Q}_e - \Phi_m \mathcal{Q}_m] \\ & - 2\pi\sqrt{z+4} |\Omega_{\text{Nil}}|. \end{aligned}$$

However, in this case, we can not use a duality argument in order to properly fix the magnetic potential. Hence, the magnetic potential and charge will only be defined up to two constants A and B , and the thermodynamics quantities read off from (21) are given by

$$\begin{aligned} \mathcal{M} = & 2|\Omega_{\text{Nil}}| \sqrt{z+4} m r_h^{z+4}, \quad T = \frac{(2z-mz+4)r_h^z}{4\pi}, \\ \mathcal{S} = & 2\pi|\Omega_{\text{Nil}}| \sqrt{z+4} r_h^4, \quad \Phi_e = \sqrt{\frac{2(m-1)}{z}} r_h^2, \\ \mathcal{Q}_e = & |\Omega_{\text{Nil}}| \sqrt{2z(z+4)(m-1)} r_h^{z+2}, \\ \Phi_m = & A\sqrt{m-1} r_h^2, \quad \mathcal{Q}_m = |\Omega_{\text{Nil}}| B\sqrt{m-1} r_h^{z+2}, \\ AB = & 2\sqrt{z+4}. \end{aligned} \quad (31)$$

Nevertheless, in spite of the ‘‘arbitrariness’’ concerning the magnetic potential and charge, the first law of thermodynamics only requires the product of the constants A and B , and it is a matter of check to see that the first law (26) effectively holds, and just need that $AB = 2\sqrt{z+4}$.

V. DYONIC SOLUTIONS WITH A HYPERSCALING VIOLATION FACTOR

One of the main interest in extending the adS/CFT correspondence to other areas of the physics was precisely to have a better understanding of strongly coupled systems of the condensed matter physics. In condensed matter physics, the notion of quantum phase transition is of great importance and it occurs at some critical point where the system may display a hyperscaling violation reflected by the fact that the entropy does not scale with its spatial

dimensionality. From the gravity side, such hyperscaling violating systems can be described by the so-called hyperscaling violating metrics [35] which are conformally related to the Lifshitz metric as

$$ds^2 = \frac{1}{r^{\frac{2\theta}{D-2}}} \left[-r^{2z} dt^2 + \frac{dr^2}{r^2} + r^2 \sum_{i=1}^{D-2} dx_i^2 \right], \quad (32)$$

in such a way that the anisotropic transformations (2) act now as a conformal transformation, i.e., $ds^2 \rightarrow \lambda^{\frac{2\theta}{D-2}} ds^2$. Here the parameter θ is the so-called the *hyperscaling violation factor* responsible of the violation of the hyperscaling property. Of course, hyperscaling violation black holes refer to black hole solutions whose asymptotic forms match with the metric (32), see, e.g., [36–39]. As before, this notion of hyperscaling violation black holes can be enlarged by relaxing the fact that the topology of the horizon is flat but still requiring that the generalized dilatation transformations (4) act as a conformal transformation for the metric solution in the asymptotic region. In this context, a hyperscaling violation black hole of general relativity was found in [23] where the horizon topology is modeled by the Nil's geometry and where the dynamical exponents are $z = \frac{3}{2}$ and $\theta = \frac{9}{2}$. As done previously, we will see that this vacuum solution can be electromagnetically charged by turning on a dilatonic source. For this purpose, we consider a slightly different action than (12)

$$S = \int d^{\bar{D}}x \sqrt{-g} \left[\frac{R}{2} - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - U(\phi) - \sum_{i=1}^N \frac{1}{4} e^{\lambda_i \phi} F_{(i)\mu\nu} F_{(i)}^{\mu\nu} \right], \quad (33)$$

where the potential is

$$U(\phi) = \Lambda e^{\gamma \phi}. \quad (34)$$

The equations of motions read

$$G_{\mu\nu} = T_{\mu\nu} - g_{\mu\nu} U(\phi), \quad (35a)$$

$$\nabla_\mu (e^{\lambda_i \phi} F_{(i)}^{\mu\nu}) = 0, \quad (35b)$$

$$\square \phi = \sum_{i=1}^N \left(\frac{\lambda_i}{4} e^{\lambda_i \phi} F_{(i)\sigma\rho} F_{(i)}^{\sigma\rho} \right) + \frac{dU}{d\phi}, \quad (35c)$$

where the energy-momentum tensor $T_{\mu\nu}$ is given by (14). In what follows, we will present two dyonic solutions of the field equations (35) with Solv and Nil's horizon topologies.

A. Hyperscaling violation dyonic Solv's solution

We first report a solution of the field equations (35) where the event horizon is given by the Solv's geometry (5a). The metric element and fields are given by

$$\begin{aligned} ds^2 &= \frac{1}{r^{\frac{2\theta}{3}}} \left[-r^{2z} f(r) dt^2 + \frac{1}{r^2 f(r)} dr^2 + r^2 x_3^2 dx_1^2 \right. \\ &\quad \left. + r^2 \frac{dx_2^2}{x_3^2} + \left(\frac{2}{z+2-\theta} \right) \frac{dx_3^2}{x_3^2} \right], \\ f(r) &= 1 - m \left(\frac{r_h}{r} \right)^{z+2-\theta} + (m-1) \left(\frac{r_h}{r} \right)^{2z+2-2\theta}, \\ F_{(1)rt} &= \sqrt{(z+2-\theta)(z-1)} r^{z+1-\theta}, \\ F_{(2)rt} &= \sqrt{(z-\theta)(1-\theta)(m-1)} \left(\frac{r_h}{r} \right)^{z+1-\theta}, \\ F_{(3)x_1x_2} &= \sqrt{(z-\theta)(m-1)} r_h^{z+1-\theta}, \\ e^\phi &= r \sqrt{2(z-1) - \frac{\theta}{3}(3z-\theta)}, \end{aligned} \quad (36)$$

provided that the coupling are tied as follows

$$\begin{aligned} \Lambda &= -\frac{(z+2-\theta)^2}{2}, \quad \gamma = \frac{2\theta}{\sqrt{18(z-1) - 3\theta(3z-\theta)}}, \\ \lambda_1 &= -\frac{4(3-\theta)}{\sqrt{18(z-1) - 3\theta(3z-\theta)}}, \\ \lambda_2 &= \frac{2(3z-3-\theta)}{\sqrt{18(z-1) - 3\theta(3z-\theta)}}, \\ \lambda_3 &= -\frac{2(3z-3-2\theta)}{\sqrt{18(z-1) - 3\theta(3z-\theta)}}. \end{aligned}$$

It is interesting to note that in this limiting case $\theta \rightarrow 0$, the constant γ goes to zero and hence the potential term (34) becomes a cosmological constant term. Consequently, in the absence of the hyperscaling violation factor $\theta = 0$, the solution reduces to the dyonic solution found previously, see Sec. III.

As before, in order to provide a complete thermodynamics analysis of the Solv's solution with hyperscaling violation, we opt for the Hamiltonian formalism where the reduced action (24) becomes

$$\begin{aligned} I_E &= \beta |\Omega_{\text{Solv}}| \int_{r_h}^{\infty} \left(N \mathcal{H} + \sum_{i=1}^2 A_{(i)\tau} \bar{\mathcal{P}}'_{(i)} \right) dr + B_E, \\ \mathcal{H} &= \sqrt{\frac{2}{z+2-\theta}} \left[r^{2-\theta} \left(1 + \frac{z-\theta}{2} \right) + \frac{1}{r^\theta} F \left(1 + \frac{\theta^2 - 3\theta}{3} \right) \right. \\ &\quad \left. + r^{1-\theta} F' \left(1 - \frac{\theta}{2} \right) + \frac{r^{2-\theta}}{2} F(\phi')^2 \right. \\ &\quad \left. + \sum_{i=1}^2 \frac{(z+2-\theta)}{4r^{2-\frac{\theta}{3}}} e^{-\lambda_i \phi} \bar{\mathcal{P}}_{(i)}^2 + \frac{e^{\lambda_3 \phi}}{2r^{2+\frac{\theta}{3}}} (F_{(3)x_1x_2})^2 \right], \end{aligned} \quad (37)$$

and where the conjugate momenta are given by

$$\bar{\mathcal{P}}_{(i)} = \sqrt{\frac{2}{z+2-\theta}} \frac{r^{2-\frac{\theta}{3}}}{N} e^{\lambda_i \phi} \partial_r A_{(i)\tau}, \quad i = 1, 2.$$

The variation of the boundary term yields

$$\delta B_E = \beta |\Omega_{\text{Solv}}| \left\{ \sqrt{\frac{2}{z+2-\theta}} \left[\delta \left(\left(1 - \frac{\theta}{2}\right) m r_h^{z+2-\theta} \right) - \frac{2\pi}{\beta} \delta r_h^{2-\theta} \right] - \Phi_e \delta \left(\sqrt{\frac{2(z-\theta)(1-\theta)(m-1)}{z+2-\theta}} r_h^{z+1-\theta} \right) - \sqrt{\frac{2(m-1)}{z+2-\theta}} r_h \delta(\sqrt{m-1} r_h^{z+1-\theta}) \right\},$$

with

$$\Phi_e = \sqrt{\frac{(1-\theta)(m-1)}{(z-\theta)}} r_h.$$

After some computations, we finally conclude that the thermodynamical quantities are

$$\begin{aligned} \mathcal{M} &= |\Omega_{\text{Solv}}| \sqrt{\frac{2}{z+2-\theta}} \left(1 - \frac{\theta}{2}\right) m r_h^{z+2-\theta}, \\ T &= \frac{[(m-2)\theta - z(m-2) + 2] r_h^z}{4\pi}, \\ \mathcal{S} &= 2\pi |\Omega_{\text{Solv}}| \sqrt{\frac{2}{z+2-\theta}} r_h^{2-\theta}, \\ \Phi_e &= \sqrt{\frac{(1-\theta)(m-1)}{(z-\theta)}} r_h, \\ \mathcal{Q}_e &= |\Omega_{\text{Solv}}| \sqrt{\frac{2(z-\theta)(1-\theta)(m-1)}{z+2-\theta}} r_h^{z+1-\theta}, \\ \Phi_m &= A \sqrt{m-1} r_h, \\ \mathcal{Q}_m &= |\Omega_{\text{Solv}}| B \sqrt{m-1} r_h^{z+1-\theta}, \quad AB = \sqrt{\frac{2}{z+2-\theta}}, \end{aligned} \quad (48)$$

and we check again the validity of the first law (26).

To end this section, we would like to point out an interesting observation. For the Solv's solution without hyperscaling violation parameter, we have shown that the charged solution must necessarily be electric and magnetic. Here, the presence of the hyperscaling factor θ allows to switch off the electric contribution putting $\theta = 1$, and the resulting configuration turns to be purely magnetic.

B. Hyperscaling violation black hole with a Nil's geometry

We now turn to the construction of the dyonic extension of the vacuum Nil's solution [23] which is given by

$$\begin{aligned} ds^2 &= \frac{1}{r^{\frac{2\theta}{3}}} \left[-r^{2z} f(r) dt^2 + \frac{1}{r^2 f(r)} dr^2 + r^2 dx_1^2 + r^2 dx_2^2 + (z+4-\theta) r^4 (dx_3 - x_1 dx_2)^2 \right], \\ f(r) &= 1 - m \left(\frac{r_h}{r}\right)^{z+4-\theta} + (m-1) \left(\frac{r_h}{r}\right)^{2z+4-2\theta}, \\ e^\phi &= r \sqrt{2(2z-3) - \frac{\theta}{3}(3z-\theta)}, \\ F_{(1)rt} &= \frac{\sqrt{2(2z-3)(z+4-\theta)}}{2} r^{z+3-\theta}, \\ F_{(2)rt} &= \frac{\sqrt{(2-\theta)(z-\theta)(m-1)} r_h^{z+2-\theta}}{r^{z+1-\theta}}, \\ F_{(3)x_1x_2} &= -x_1 \sqrt{(z-\theta)(m-1)(z+4-\theta)} r_h^{z+2-\theta}, \\ F_{(3)x_1x_3} &= \sqrt{(z-\theta)(m-1)(z+4-\theta)} r_h^{z+2-\theta}, \\ F_{(4)x_2x_3} &= \sqrt{(z-\theta)(m-1)(z+4-\theta)} r_h^{z+2-\theta}, \end{aligned} \quad (39)$$

provided that

$$\begin{aligned} \Lambda &= -\frac{(z+4-\theta)(z+3-\theta)}{2}, \\ \lambda_1 &= -\frac{4(6-\theta)}{\sqrt{18(2z-3) - 3\theta(3z-\theta)}}, \\ \lambda_2 &= \frac{2(3z-6-\theta)}{\sqrt{18(2z-3) - 3\theta(3z-\theta)}}, \\ \lambda_3 = \lambda_4 &= -\frac{2(3z-2\theta-3)}{\sqrt{18(2z-3) - 3\theta(3z-\theta)}}, \end{aligned}$$

and the Liouville coupling potential (34) is given by

$$\gamma = \frac{2\theta}{\sqrt{18(2z-3) - 3\theta(3z-\theta)}}.$$

In the limiting case, $z = 3/2$, $\theta = 9/2$ and $m = 1$, one effectively recovers the vacuum solution found in [23]. Also, note that for a hyperscaling violation factor $\theta = 2$, the solution can be rendered purely magnetic. Now, proceeding as before, one obtains the following thermodynamical quantities

$$\begin{aligned}
\mathcal{M} &= |\Omega_{\text{Nil}}| \sqrt{z+4-\theta} \left(2 - \frac{\theta}{2}\right) m r_h^{z+4-\theta}, \\
T &= \frac{[(m-2)\theta + z(2-m) + 4] r_h^z}{4\pi}, \\
\mathcal{S} &= 2\pi |\Omega_{\text{Nil}}| \sqrt{z+4-\theta} r_h^{4-\theta}, \\
\Phi_e &= \sqrt{\frac{(2-\theta)(m-1)}{z-\theta}} r_h^2, \\
\mathcal{Q}_e &= |\Omega_{\text{Nil}}| \sqrt{(z+4-\theta)(2-\theta)(m-1)(z-\theta)} r_h^{z+2-\theta}, \\
\Phi_m &= A \sqrt{m-1} r_h^2, \\
\mathcal{Q}_m &= |\Omega_{\text{Nil}}| B \sqrt{m-1} r_h^{z+2-\theta}, \quad AB = 2\sqrt{z+4-\theta},
\end{aligned} \tag{40}$$

which satisfy the electromagnetic version of the first law of thermodynamics (26).

VI. CONCLUSIONS

Here, we have shown that the vacuum solutions with Solv and Nil's horizon topologies of the five-dimensional Einstein equations can be electromagnetically charged through a dilatonic source with at least three Abelian gauge fields. The resulting dyonic solutions are asymptotically anisotropic and can be considered as Lifshitz black holes in the sense as defined by Eqs. (3)–(4). The presence of various Abelian fields is mandatory in order to ensure the Lifshitz asymptotic and the emergence of the electric and magnetic charges. Through an Hamiltonian approach, we have realized a complete analysis of the thermodynamics features of the dyonic solutions, and we have checked that for each solution, the electromagnetic version of the first law of thermodynamics is satisfied. We have noticed that the dyonic solutions, in spite of having a Reissner-Nordstrom-like metric, are quite different from the magnetically charged Reissner-Nordstrom solution. Indeed, for the dyonic Lifshitz solutions with Thurston's horizon

topologies, the electric and magnetic charges turn to be proportional. In other words, there does not exist a purely electric or purely magnetic solution. This characteristic is similar to what occur for the odd-dimensional Chern-Simons vortices (see [40] for a good review). Indeed, because of the presence of the three-dimensional Chern-Simons term $\kappa \epsilon^{\mu\nu\rho} A_\mu F_{\nu\rho}$ in the action, the magnetically charged vortices also carry an electric charge proportional to the magnetic charge. It is important to stress again that the presence of the dyonic charges allows the Lifshitz dynamical exponent to be free and not restricted as in the vacuum case. Such a feature was already observed in [17] where the presence of a nonlinear electrodynamics source was responsible of the freedom of the dynamical exponent.

The hyperscaling violation extensions of these dyonic solutions were also considered. In this case, the dilatonic source is augmented by a Liouville potential term and the cosmological constant is turned off. We have noticed that for some specific values of the hyperscaling violation factor, the dyonic solutions can be rendered purely magnetic.

An interesting work to be done will consist in computing for the dyonic solutions reported here the DC conductivities of the corresponding field theory in order to gain some precision about this latter, see, e.g., [32]. Also, very recently, a new dyonic solution of the Einstein-Maxwell-dilaton theory was constructed in [41] using some solution-generating technique. It will be interesting to see whether these techniques can be exported in our problem to generate new dyonic solutions.

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