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Colored-Edge Graph Approach for the Modeling of Multimodal Transportation Systems

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Many networked systems involve multiple modes of transport. Such systems are called multimodal, and examples include logistic networks, biomedical phenomena and telecommunication networks. Existing techniques for determining minimal paths in multimodal networks have either required heuristics or else application-specific constraints to obtain tractable problems, removing the multimodal traits of the network during analysis. In this paper weighted colored-edge graphs are introduced for modeling multimodal networks, where colors represent the modes of transportation. Minimal paths are selected using a partial order that compares the weights in each color, resulting in a Pareto set of minimal paths. Although the computation of minimal paths is theoretically intractable and $\mathcal{NP}$-complete, the approach is shown to be tractable through experimental analyses without the need to apply heuristics or constraints.

Keywords: Multimodal transportation system; shortest path; weighted colored-edge graph.

1. Introduction

Extensive scientific literature has been devoted in the last three decades to the study of multimodal networks (MMN). During this time, research has mainly focused on practical applications for freight or urban transportation, extensive reviews are found in Jarzemskiene (2007) and Macharis and Bontekoning (2004).

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As a system in which several means of transport are available, a multimodal system is able to emulate a wide spectrum of real life phenomena beyond the field of logistics. Areas such as computer networks, biology and manufacturing have begun to utilize multimodal networks for studying and modeling situations. Examples can be found in papers by Abrach *et al.* (2003), Chen *et al.* (2005), Heath and Sioson (2007), Kiesmüller *et al.* (2005), Nigay and Coutaz (1993), Medeiros *et al.* (2000) and Sioson (2005).

From a modeling perspective, a range of techniques have been used to model MMN. They can be loosely classified into three predominant domains: mathematical programming, weighted graphs and multi-weighted graphs.

1.1. *The mathematical programming approach*

These techniques are characterized by making use of linear or nonlinear formulations for representing a MMN by a set of equations.

Linear programming techniques are suitable when each decision variable is a linear combination of the problem parameters, Hillier and Lieberman (2009). Integer programming and mixed integer programming stand out as the most common linear programming techniques used for multimodal modeling. Sample papers using linear programming as a modeling tool are given by Min (1991) and Kim *et al.* (1999).

Nonlinear programming is another renowned modeling technique for MMN. It is mainly used to build intricate cost functions, and principally deals with second order equations satisfying convex or concave properties. Examples are provided by Kim and Kim (2006), Horn (2003) and Chang (2007) which have opted to use nonlinear programming as their main modeling approach.

In the mathematical programming approach, mode options are visualized as decision variable indices, which considerably increases the complexity of the problem. Relaxation or cutting plane techniques are commonly used to make the problem tractable. Interesting papers tackling general views of mathematical programming for intermodal transportation (the transportation of goods) and urban transportation are presented by Jarzemskiene (2007) and Nagurney (1984), respectively.

1.2. *The weighted graph approach*

In this approach, a node typically represents a location, such as a warehouse, transportation hub or network router, and an edge represents a transportation link, such as a rail line, a bus or a wireless connection. A variety of graphs have been used to study these transport systems, such as digraphs, multigraphs, hypergraphs and grid graphs. Ayed *et al.* (2008) provides a general classification for MMN models based on weighted graph approaches. In particular, the paper emphasizes the use of multigraphs, in which there might be multiple edges between two nodes, and the use of grids in which a grid is overlayed on a planimetric map. Both can result in dense graphs, which require edge reduction techniques to make their analysis tractable.

In practice such reductions rely on enforcing constraints on feasible edges in order to build a specific path. In the context of freight transportation, such constraints usually correspond to timetables, consolidation options, time windows, scheduling options and suchlike. Studies making use of such graphs and constraints are provided by Foo *et al.* (1999), Qiang Li (2000), Kitamura *et al.* (1999), Fragouli and Delis (2002) and Moccia *et al.* (2011). Hypergraphs are another type of graph used in some articles. In graph theory, a hypergraph is a generalization of a graph, where edges can connect any number of vertices (Lawler, 2001). In the multimodal context, such graphs have found interesting applications in biology and urban transportation. Sample papers using hypergraphs to represent MMN are yielded by Heath and Sioson (2007), Sioson (2005) and Lozano and Storchi (2002). A recent work using multilevel graphs for modeling urban transportation networks is presented by Ma (2014). A A* label-setting algorithm is designed by the authors for solving the model.

In effect, the weighted graph approach only utilizes mode information during the application of constraints, removing the multimodal traits from the network during modeling. The analysis in this approach is very application-dependent as it relies on applying application-specific constraints.

1.3. *The multi-weighted graph approach*

This approach has been extensively utilized for the multicriteria shortest path problem (MSPP) which has become a fruitful branch of research since the 1980s, see Tarapata (2007) and Clímaco and Pascoal (2012) for a complete review. Basically, the approach assigns multiple weights to each edge. In particular, the bicriteria shortest path problem assigns two weights to each edge, such as cost and time. A complete survey on solution strategies for bicriteria shortest path problems is provided by Raith and Ehrgott (2009).

Optimality in the multi-weighted graph approach is commonly established by the use of a partial order relation which results in a Pareto set of minimal paths that are candidates for the sought shortest path. There is little literature that directly applies multi-weighted graphs for modeling MMN, but the goal of MSPP is essentially the same as for the shortest path problem in MMN. Although articles developing MSPP formulations for MMN can be identified, they preferentially use partial orders to optimize route choice decisions mainly based on cost and time, leaving the mode options as an outcome of the optimal route.

In terms of complexity, the tractability of the MSPP is inextricably connected with the cardinality of the Pareto set according to Müller-Hannemann and Weihe (2006). These authors also mention an important fact: No algorithmic consideration is capable of providing a significant improvement on the tractability, so that approaches must focus on identifying and exploiting key characteristics that occur in an specific application. Garey and Johnson (1979) shows the MSPP in general is an $\mathcal{NP}$-hard problem by using a reduction to the satisfiability problem. Hansen (1980) shows that the Pareto set cardinality is exponential in the worst case by constructing a group of graphs where all source–destination paths are Pareto efficient. As a consequence, constraints are applied during analysis to make the problem tractable, resulting in a Pareto set with manageable cardinality. MMN models whose mainstay is a multi-weighted graph can be found in manuscripts by Androutsopoulos and Zografos (2009), Aifadopoulou *et al.* (2007) and Modesti and Sciomachen (1998).

This work aims to study a modeling technique capable of keeping modes throughout the analysis. The approach is tested by analyzing the computation of the shortest path problem in two cases: randomly generated MMN and a real MMN from Europe.

The paper is organized as follows. In Sec. 2, a fourth approach for modeling MMN is introduced and compared with the three previous approaches. An algorithm is described in Sec. 3 that is used in the paper to compute minimal paths in weighted colored-edge graphs. In Sec. 4, the computation of minimal paths in colored-edge graphs is proved to be both intractable and $\mathcal{NP}$-complete. The model is experimentally studied in Sec. 5. Moreover, the algorithm is applied to a real multimodal network in order to assess its practical applicability. Finally, Sec. 5 provides conclusions about this paper.

2. Weighted Colored-Edge Graph Approach

All the approaches described in Sec. 1 are heavily application specific and do not actually utilize the multimodal nature of a network. In this paper, an approach to model and analyze multimodal networks is introduced. In essence, such approach uses a weighted graph in which edges are endowed with two attributes: a positive weight and a discrete variable called color.

A weighted colored-edge graph $G = (V, E, \omega, \lambda)$ consists of a directed graph (V, E) with vertex set V and edge set E, a weight function $\omega\colon E \to \mathbb{R}^+$ on edges, and a color function $\lambda\colon E \to M$ on edges, where M is a set of colors. Typically M is taken as a finite set with $k = |M|$. Associated to each edge $e_{uv} \in E$ from vertex u to vertex v there is a positive weight $\omega(e_{uv})$ and a color $\lambda(e_{uv}) = c$. For any color $i \in M$ and for any path $p_{uv} = \{e_{x_0x_1}, e_{x_1x_2}, e_{x_2x_3}, \ldots, e_{x_{l-1}x_l}\}$ between two vertices $u = x_0$ and $v = x_l$, where each $x_i \in V$, the path weight $\omega_c(p_{uv})$ in color c is defined as $\omega_c(p_{uv}) = \sum_{e_{x_ix_{i+1}} \in p,\, \lambda(e_{x_ix_{i+1}})=c} \omega(e_{x_ix_{i+1}})$. The total path weight is represented as a k-tuple $(\omega_{c_1}(p_{uv}), \ldots, \omega_{c_k}(p_{uv}))$, giving the total weight of the path in each color.

Note that there is no restriction placed on the number of edges e_{uv} from a vertex u to a vertex v. However, in practice for the shortest path problem attention can be restricted to weighted colored-edge graphs for which there is at most one edge e_{uv} in each color from u to v.

Let u and v be two given vertices of G and let $\mathcal{P}_{uv}$ be the set of all paths from u (source) to v (destination) in G. A binary relation between two paths p_{uv} and p'_{uv}, is defined by $p_{uv} \leq p'_{uv}$ if and only if $\omega_c(p_{uv}) \leq \omega_c(p'_{uv})$ for all c. The relation

$\leq$ is clearly reflexive, transitive and antisymmetric and gives a partial order on the k-tuple path weights, but only a preorder on the paths themselves as multiple paths might have the same total path weight.

Let $\mathcal{M}_{uv} = \{p_{uv} \in \mathcal{P}_{uv} \mid \forall\, p'_{uv} \in \mathcal{P}_{uv}$ with $\omega(p'_{uv}) \neq \omega(p_{uv}), \exists$ color c such that $\omega_c(p_{uv}) < \omega_c(p'_{uv})\}$ be the set of Pareto minimal paths joining vertices u and v. This set has an important characteristic: for any $p_{uv} \in \mathcal{M}_{uv}$, it is impossible to determine a path p'_{uv} from u to v which has smaller weight than p_{uv} in some of its k colors without at least one of the other weights being larger, analogously to Martins (1984).

From the above definitions, it is apparent that the concept of a weighted colored-edge graph with k colors can equivalently be formulated as a multiweighted multigraph where each edge is assigned a k-tuple of non-negative weights $(w_{c_1}, \ldots, w_{c_i}, \ldots, w_{c_k})$ and exactly one $w_{c_i} > 0$. However, multiweighted graphs are mostly used in multicriteria optimization applications where the weight components correspond to quantities to be optimized, such as cost and time, so edges typically contribute toward more than just one quantity. For this reason, multiweighted graphs whose edge weights are zero in all but one component have not received attention in the literature.

Shortest path analysis in the weighted colored-edge graph approach is seen in this paper to typically be tractable without the need to apply any application-specific heuristics or constraints, so can be considered a general tool for the study of multimodal networks. Application-specific considerations can still be applied to the resulting set $\mathcal{M}_{uv}$, or a post-optimal analysis undertaken on it. One facet of this model is that it can be directly applied to multigraph applications, such as transportation networks where there are multiple transportation means between two locations, communication networks where there are multiple links or choice of communication protocols between nodes, or epidemic models which have multiple paths of infection.

However, focusing attention only on the Pareto minimal paths limits the approach to shortest path problems where just the summed contribution of each color is important, and where any measure of optimality is presumed to be an increasing (linear or nonlinear) function of the summed contribution in each color. For instance, the approach presumes in a transportation network that the minimal path (such as least cost, time, or distance) is some application-specific increasing function of the total weight in each transportation means, or that the user can apply some application-specific criteria to select a preferred path from the Pareto set once it has been determined.

The approach can be also adapted for path constraints such as restricting the number of hops or the number of mode changes by slightly enhancing the algorithm used to determine the Pareto set. For instance, besides using colors to represent the different transportation means, an additional color can be used to count the number of edges in a path as the path is being built during the analysis and/or to count the number of transfers from one means of transportation to another.

Optimization problems that utilize models similar to weighted colored-edge graphs have received little attention in the literature. Climaco *et al.* (2010) experimentally studied the number of spanning trees in a weighted graph whose edges are labeled with a color. In that work, weight and number of colors are two criteria to be both minimized and the proposed algorithm generates a set of nondominated spanning trees. In a similar work, Pascoal *et al.* (2013) develop an algorithm to generate a set of nondominated paths. Edges in the graph have two associated criteria: a cost value and a label (color). Unlike the previous work which tackles spanning trees, the authors focus on analyzing paths with two attributes. The computation of colored paths in a weighted colored-edge graph is investigated by Xu *et al.* (2009). The main feature of their approach is a graph reduction technique based on a priority rule. This rule basically transforms a weighted colored-edge multidigraph into a colored-vertex digraph by applying algebraic operations to the adjacency matrix. Additionally, the authors provide an algorithm to identify colored source–destination paths. Nevertheless, the algorithm is not intended for general instances because its input is a unit weighted colored multidigraph and only paths not having consecutive edges equally colored are considered.

This paper investigates the feasibility of the colored-edge graph as a general tool for multimodal transportation systems by determining the cardinality of the Pareto set $\mathcal{M}_{uv}$ for many randomly weighted networks. Despite the intractability and $\mathcal{NP}$-completeness of the problem (Sec. 4), the cardinality is shown to typically be a very low order polynomial function of the size of the network, and demonstrates that even dense multimodal graphs with hundreds of thousands of edges can be feasibly analysed using this approach, without the need for any reduction techniques. In fact, it is seen that the number of modes k is more of a limiting factor of the approach than is the number of vertices or edges in the graph.

3. Algorithm for Determining Pareto Set of Minimal Paths

To experimentally study the feasibility of using weighted colored-edge graphs for multimodal networks an algorithm that determines $\mathcal{M}_{uv}$ is required. Since no contribution from an algorithmic view is sought by this manuscript, a simple generalization of Dijkstra's algorithm from unimodal networks has been developed for the purposes of this research, although more efficient algorithms might be investigated in the future.

The classic Dijkstra's algorithm for solving the single-source shortest path problem in unimodal networks uses a priority queue Q to store shortest path estimates from a fixed source vertex s to each vertex v in the network until the shortest path to v is determined. Since the weights of any paths p_{sv} from s to v are linearly ordered there is only at most one shortest path estimate in the queue at a time for each vertex v. At the start of each iteration of the algorithm the shortest path estimate at the front of the queue is the actual shortest path to one of the vertices in the network.

In a weighted colored-edge graph Dijkstra's algorithm must be slightly generalised to handle weights of paths being partially ordered rather than linearly ordered. A priority queue Q can again be used to store shortest path estimates with the requirement that if a path p_{sv} from s to v has smaller weight than another path p'_{sv} then it must appear earlier in the queue. Although the results presented in this paper use such a simple queue instead of a more sophisticated nonlinear data structure the performance of the algorithm is seen to be surprisingly good for colored-edge graphs with random weights. As in the classical Dijkstra's algorithm the weighted colored-edge version of the algorithm takes as input a network G and a source vertex s. It commences at s with the empty path p_{ss} and relaxes each edge that is incident from the source vertex s, adding the single edge paths to the queue. At the front of the queue will be a shortest path estimate p_{sv} to some vertex v adjacent to s. Since all weights are positive in the network p_{sv} must have minimal weight amongst paths from s to v (although it might not be the only minimal path from s to v in the queue), so p_{sv} is added to the set $\mathcal{M}_{sv}$ and removed from the queue. The algorithm then relaxes all the edges incident to v, extending the path p_{sv} by each edge to a path $p'_{su} = p_{sv} \cup \{e_{vu}\}$, adding those extended paths p'_{su} to the queue that have minimal weight amongst paths from s to u, and removing from the queue any path p''_{su} from s to u that has greater weight than p'_{su}. The algorithm repeats itself until the queue is empty, producing as output the Pareto set $\mathcal{M}_{sv}$ for each vertex v in the network. Figure 1 describes the pseudocode of the algorithm using the notation developed by Cormen *et al.* (2001).

Multimodal-Dijkstra(G, s)

▷ Initially no Pareto minimal paths known
for each vertex v
    do $\mathcal{M}_{sv} \leftarrow \emptyset$
▷ Create a queue Q to hold shortest path estimates during processing
$Q \leftarrow \emptyset$
add the empty path p_{ss} from s to s into Q
while $Q \neq \emptyset$
    do remove the path p_{sv} at front of Q that has some end vertex v
        add the path p_{sv} to $\mathcal{M}_{sv}$
        ▷ Relax the edges incident from v
        for each edge e_{vu} incident from v to a vertex u not in p_{sv}
            do ▷ Extend the path p_{sv} by the edge e_{vu}
                $p'_{su} \leftarrow p_{sv} \cup \{e_{vu}\}$
                if p'_{su} has minimal weight in Q from s to u
                    then add p'_{su} to Q
                        ▷ Remove any paths no longer minimal in Q
                        for each $p''_{su} \in Q$ with $\omega\left(p''_{su}\right) > \omega\left(p'_{su}\right)$
                            do remove the path p''_{su} from Q
return $\mathcal{M}_{sv}$

Fig. 1. Pseudocode of the algorithm.

The iteration of this algorithm works same as Martins' algorithm (see Martins, 1984). The approach proposed by Martins is a label setting algorithm for the multicriteria shortest path problem. At each iteration, two sets of labels are estimated: permanent and temporary. Martins's approach also produce a set of minimal paths. Analogously, the multimodal Dijkstra's algorithm uses the queue datastructure to manage "temporary" paths estimates that could become "permanent" or part of $\mathcal{M}_{sv}$ when the partial order relation is applied.

The number of relaxation steps is an important indicator of the algorithm's order, so besides finding $\mathcal{M}_{sv}$ the experiment discussed in Sec. 5 also tracks the number of paths p'_{su} processed by the algorithm.

As an example of an application of the weighted colored-edge graph approach, the algorithm is run with a multimodal network developed by Lozano and Storchi (2001) starting at source vertex 0. Figure 2 shows the network which has 21 vertices, 51 edges and 4 different transport choices (bus, metro, private and transfer).

The algorithm commences with just the empty path p_{00} on the queue and relaxes two edges: e_{01} with weight $(\text{bus}, \text{metro}, \text{private}, \text{transfer}) = (15, 0, 0, 0)$, and e_{03} with weight $(0, 0, 5, 0)$, which are both added to the queue. Since the two weights are incomparable, either could be at the front of the queue, so the next iteration of the algorithm either adds the path $p_{01} = \{e_{01}\}$ to $\mathcal{M}_{01}$ and relaxes the four edges incident to vertex 1 by extending the path p_{01} by each, or else adds the path $p_{03} = \{e_{03}\}$ to $\mathcal{M}_{03}$ and relaxes the three edges incident to vertex 3 by extending the path p_{03} by each. Continuing in this way the Pareto set $\mathcal{M}_{0v}$ is obtained for each vertex v in the network, resulting in 52 Pareto minimal paths from vertex 0 to vertex 20 whose weights are listed in Table 1.

Depending on the application, constraints or heuristics can then be applied to the 52 paths to select a path preferred by the user. Using just a simple priority queue data structure the generalized Dijkstra's algorithm can determine $\mathcal{M}_{0v}$ for all 21 vertices v within approximately 10 ms. The article by Lozano and Storchi (2001) instead uses a weighted graph approach with application-specific constraints and a simple cost function which adds the weights in each mode together to get a single valued total weight, resulting in the paths numbered 2, 25, 33, 47 in the table.

Note that a Pareto set permits a post-optimal analysis to be carried out provided that the total cost is presumed to be an increasing 4-ary function[a] of the summed weight in each mode. For instance, suppose in the example network that the edge weights represent distance and for simplicity presume the cost is one dollar per unit distance for each means of transport. A natural optimization question could be how much the unit cost associated to a particular mode could be increased or decreased with the current minimal path remaining minimal. As an illustration, path 25 which has the edges $\{e_{03}, e_{31}, e_{19}, e_{9\,10}, e_{10\,14}, e_{14\,15}, e_{15\,17}, e_{17\,16}, e_{16\,18}, e_{18\,19}, e_{19\,20}\}$

[a] A k-ary function is a function that takes k arguments as input. In the context of this article, these arguments correspond to the total weight in each mode that a path can generate.

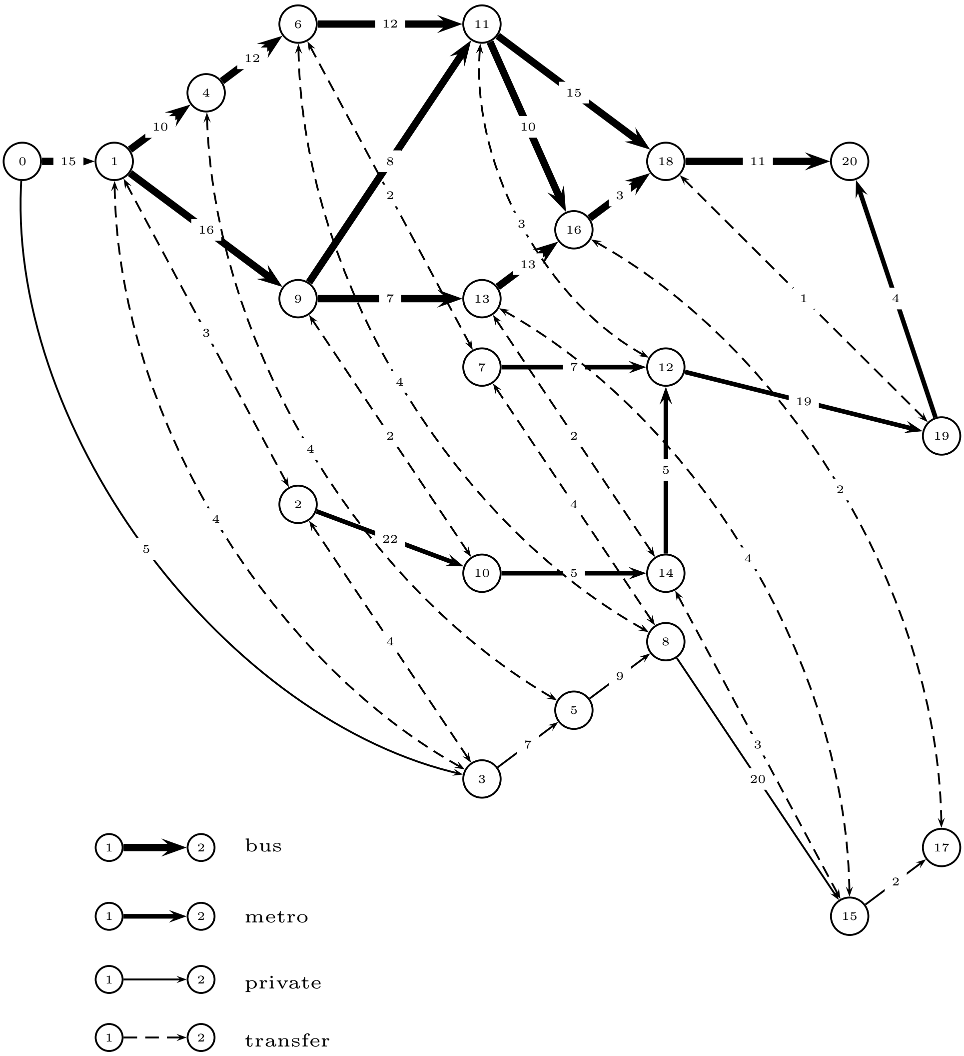

Fig. 2. Multimodal network from paper Lozano and Storchi (2001).

and shown in Fig. 3 has least total cost \$47, but from the Pareto set it is easily seen that an increase of over 20% in the relative metro costing would make path 41 with edges $\{e_{03}, e_{31}, e_{19}, e_{9\,13}, e_{13\,15}, e_{15\,17}, e_{17\,16}, e_{16\,18}, e_{18\,19}, e_{19\,20}\}$ a better choice, or a 25% increase in bus prices would make path 16 with edges $\{e_{03}, e_{32}, e_{2\,10}, e_{10\,14}, e_{14\,15}, e_{15\,17}, e_{17\,16}, e_{16\,18}, e_{18\,19}, e_{19\,20}\}$ better.

The approach can be adapted for path constraints such as restricting the number of hops or the number of mode changes by slightly enhancing the algorithm. For instance, besides using colors to represent the different transportations options, an additional color can be used to count the number of edges in a path as the path is

Table 1. Pareto set for network with 21 vertices and 51 edges.

Path number	Transport choice cost				Cost as per Lozano and Storchi (2001)
	Bus	Metro	Private	Transfer	
1	25	4	21	5	55
2	0	30	21	4	55
3	32	9	5	9	55
4	13	11	21	8	53
5	24	0	36	10	70
6	11	26	21	5	63
7	21	26	5	7	59
8	50	19	0	4	73
9	41	4	2	7	54
10	8	45	5	9	67
11	3	4	43	3	53
12	13	4	36	11	64
13	10	30	14	12	66
14	25	26	2	12	65
15	26	0	34	10	70
16	3	31	7	10	51
17	16	4	41	5	66
18	47	9	0	5	61
19	31	31	0	6	68
20	14	0	43	2	59
21	23	45	0	8	76
22	52	4	0	1	57
23	24	7	21	7	59
24	25	11	12	10	58
25	19	9	7	12	47
26	32	22	5	6	65
27	16	31	5	7	59
28	29	27	2	8	66
29	36	0	21	4	61
30	15	4	34	11	64
31	14	27	7	9	57
32	12	23	21	7	63
33	63	0	0	0	63
34	39	23	0	3	65
35	24	23	5	7	59
36	36	26	0	6	68
37	12	30	12	6	60
38	10	26	7	13	56
39	18	31	2	9	60
40	27	0	41	4	72
41	26	4	7	11	48
42	29	0	38	6	73
43	52	0	2	6	60
44	22	30	5	6	63
45	34	9	2	8	53
46	48	0	5	4	57

Table 1. (*Continued*)

Path number	Transport choice cost				Cost as per Lozano and Storchi (2001)
	Bus	Metro	Private	Transfer	
47	37	4	5	5	51
48	37	30	0	2	69
49	36	7	12	9	64
50	18	4	38	7	67
51	27	27	5	6	65
52	37	0	7	10	54

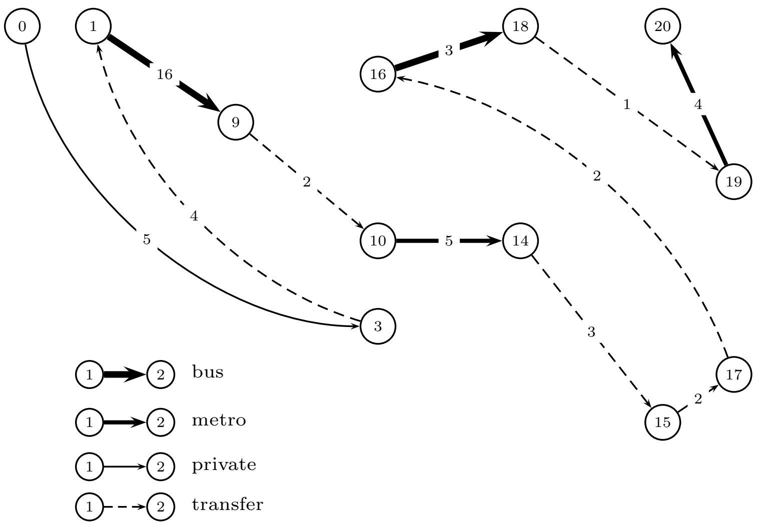

Fig. 3. Path 25 used for a post-optimal analysis.

being built during the analysis or count the number of transfers from one means of transportation to another.

This example demonstrates that the multimodal Dijkstra's algorithm can quickly calculate the Pareto set without the need to assign relative costs for the different modes. Then alternative cost functions can be evaluated on just the paths in the Pareto set or a post-optimal analysis conducted without ever having to rerun the algorithm on the network.

In summary, Lozano and Storchi (2001) develop a graph approach that identifies viable paths (paths satisfying a set of specific constraints). By doing this, the approach reduces the size of the problem and removes the multimodal traits from the network during the computation. The weighted colored-edge graph differentiates from Lozano and Storchi (2001) approach in two aspects. First, no constraint needs to be applied to compute minimal paths. Hence the concept of viable paths is needless. Second, a post-optimal analysis can be performed in such a way a user can select mode combinations from the final Pareto set based on different cost functions.

4. Model Complexity

Next the tractability and $\mathcal{NP}$-completeness of the approach are studied. An important claim from this section is the use of either a multiweighted graph or a colored-edge graph involves an equal level of complexity (both approaches are intractable and $\mathcal{NP}$-complete). As a result, the selection of one approach depends more on both an application domain and an user decision. A multiweighted graph approach is intended for minimizing two or more conflicting criteria, leaving mode combinations as a result of an optimal path, whereas a colored-edge graph approach focuses on determining minimal paths in which the contribution of each mode is explicitly computed. In other words, a user whose focus is to establish the most convenient combination of modes for travelling from a source to a destination could take more advantage of a colored-edge graph as a modeling tool.

4.1. *Tractability of the model*

This section shows the computation of minimal paths in a colored-edge graph is in the worst case an intractable problem.

Theorem 4.1. *The computation of $\mathcal{M}_{uv}$ in a weighted colored-edge graph is, in worst case, intractable, i.e., require for some problems a number of operations which grows at least exponentially with these problem's characteristics.*

Proof. It is sufficient to show there exists for each problem a family of weighted colored-edge graphs for which the cardinality of $\mathcal{M}_{uv}$ grows exponentially with the number of vertices. First, consider a colored-edge graph $\mathbf{G} = \langle V, E, \omega, \lambda \rangle$ for which the vertex set V can be ordered $v_1, v_2, \ldots, v_{n-1}, v_n$, where $|V| = n$. Moreover, this graph only has edges from v_i to v_{i+1}. The graph has k colors and a source and destination vertex given by $u = v_1$ and $v = v_n$, respectively. Note that the total number of paths from u to v is k^{n-1}. Consider now the weights of the edge $e_c \in E$ from v_i to v_{i+1} satisfy the following conditions:

(1) For all colors c, c', the edges e_c and $e_{c'}$ from v_i to v_{i+1} have the same weight, $\omega(e_c) = \omega(e_{c'}) = \omega_i$.
(2) For the weights $W = \{\omega_1, \omega_2, \ldots, \omega_{n-1}\}$ the sum function $\sum : \mathcal{P}(W)\backslash\emptyset \to \mathbb{R}^+$ is one to one. For instance, if $\{\omega_{i_1}, \omega_{i_2}, \ldots, \omega_{i_s}\}$ and $\{\omega_{j_1}, \omega_{j_2}, \ldots, \omega_{j_t}\}$ are two distinct nonempty subsets of W, then $\omega_{i_1} + \omega_{i_2} + \cdots + \omega_{i_s} \neq \omega_{j_1} + \omega_{j_2} + \cdots + \omega_{j_t}$.

Note that by Condition (1), for any path p from $u = v_1$ to $v = v_n$ the sum of all the edge weights in the path is fixed:

$$\sum_{color\ c} \omega_c(p) = \sum_{i=1}^{n-1} \omega_i.$$

Suppose p and q are two distinct paths from $u = v_1$ to $v = v_n$. Then in some color c', p and q must use different edges, so by Condition (2) $\omega_{c'}(p) \neq \omega_{c'}(q)$. Without loss of generality suppose $\omega_{c'}(p) < \omega_{c'}(q)$. As

$$\sum_{color\ c} \omega_c(p) = \sum_{color\ c} \omega_c(q),$$

it follows that there is another color c'' for which $\omega_{c''}(p) > \omega_{c''}(q)$. Hence p and q are incomparable, so all k^{n-1} paths are minimal. $\square$

An example of a weighted colored-edge graph meeting Conditions (1) and (2) is depicted by Fig. 4, where all 3^{n-1} paths from v_1 to v_n are minimal.

4.2. $\mathcal{NP}$-completeness

An important aspect of the approach is the computation of the Pareto minimal set $\mathcal{M}_{uv}$. To appreciate the difficulty of computing minimal paths in a colored-edge graph, the $\mathcal{NP}$-completeness of the problem is next studied. Recall first that the well-known bin packing problem is $\mathcal{NP}$-complete (Garey and Johnson, 1979):

Bin packing problem

Instance: A set N of n items, each with a positive integer weight ω_i for $1 \leq i \leq n$, a positive number of bins k and a positive integer bin capacity β_j, for $1 \leq j \leq k$.

Question: Can the set N be partitioned in k subsets such that for each subset, the sum of the weights ω_i in partition i is at most β_j?

By using a reduction from the bin packing problem, the determination of minimal paths in a colored-edge graph can be shown to be $\mathcal{NP}$-complete.

Restricted minimal paths in a colored-edge graph

Instance: A colored-edge graph $\mathbf{G} = \langle V, E, \omega, \lambda \rangle$ with k colors, two distinguished vertices s and t and a maximum path weight $\alpha = (\alpha_1, \ldots, \alpha_k)$.

Question: Does there exist a path from s to t with total weight $\leq \alpha$?

Theorem 4.2. *The restricted minimal path problem in a colored-edge graph is $\mathcal{NP}$-complete.*

Proof. Use a reduction from the bin packing problem. First, consider a colored-edge graph $\mathbf{G} = \langle V, E, \omega, \lambda \rangle$ for which the vertex set V can be ordered $v_1, v_2, \ldots, v_{n-1}, v_n$,

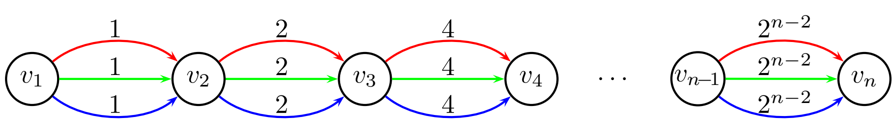

Fig. 4. A weighted colored-edge graph meeting conditions of Theorem 4.1.

where $|V| = n$, and the graph only has edges from v_i to v_{i+1}. Additionally, this graph has k colors and source and destination vertices given by $u = v_1$ and $v = v_n$, respectively. Assign each item weight ω_i to each edge from v_i to v_{i+1}. Note that each path joining u and v has a weight tuple indicating the total weight assigned to each bin. Hence, each path from u to v in a colored-edge chain is a partition that might solve the bin packing problem. Next, set $\alpha = (\beta_1, \beta_2, \ldots, \beta_k)$ and apply the condition that the weight of a path from u to v in the chain has not to be greater than α. □

The presented theorems provides a pessimistic view about the computation of minimal paths in colored-edge graphs. No polynomial time algorithm can be developed for general instances of the problem. Despite this, minimal paths are quickly computed in the next section for colored-edge graphs with random weights.

Another study of the approach complexity is presented by Ensor and Lillo (2011). The authors report here two bounds for the number of minimal paths. The paper first presents a tight upper bound for the number of minimal paths. This bound is exponential. Next, the authors show a polynomial bound for the expected number of minimal paths. A polynomial bound for colored-edge graphs with randomly generated weights indicates that the approach can be in practice very efficient.

5. Experimental Study

In this section, the weighted colored-edge graph approach is applied to multimodal networks in two different scenarios. Firstly, the cardinality of $\mathcal{M}_{uv}$ is analyzed for colored-edge graphs with random weights. Secondly, the approach is applied to a real multimodal network. This network corresponds to the transportation system of France and considers four transport choices.

5.1. *Colored-edge graphs with random weights*

The objective here is to identify general patterns for the number of processing paths and $\mathcal{M}_{uv}$ cardinality. In this test a weighted complete multigraph is taken as input so that each analytical scenario is generated by fixing values for $n = |V|$ and $k = |M|$. Such a graph is characterized by having $kn(n-1)$ edges and the maximum number of possible paths $|\mathcal{P}_{uv}| = \sum_{j=0}^{n-2} \binom{n-2}{j} k^{j+1} j!$ for $v \neq u$, which has factorial order $O(k^{n-1}(n-2)!)$. Specifically, the algorithm is run for complete multigraphs with $k = 2, 3, 4, 5$ colors and n between 20 and 200 vertices. Random edge weights are generated by means of a continuous uniform distribution of positive weights.

Figure 5 depicts the patterns followed by $\mathcal{M}_{uv}$ cardinality. The figure uses a logarithmic scale for vertical as well as horizontal axes to demonstrate the average case polynomial behavior as n increases. Table 2 provides the numerical orders determined for different k values. These results demonstrate not only that the Pareto set of minimal paths is calculated in polynomial time, but that the resulting set

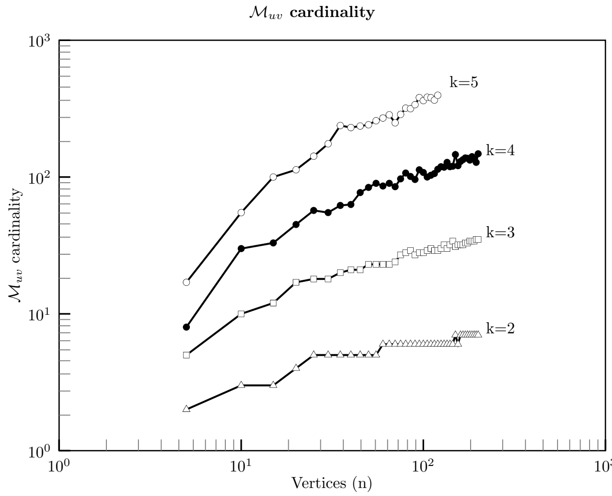

Fig. 5. Cardinality of $\mathcal{M}_{uv}$ for random weighted colored-edge graphs with different number of colors k.

Table 2. Order of processing paths and $\mathcal{M}_{uv}$ cardinality for several k values.

k	Processing paths (p'_{su})	$\mathcal{M}_{uv}$ cardinality
2	$O(n^{1.28})$	$O(n^{0.19})$
3	$O(n^{1.37})$	$O(n^{0.32})$
4	$O(n^{1.52})$	$O(n^{0.46})$
5	$O(n^{1.64})$	$O(n^{0.61})$

requiring further analysis grows very slowly as a function of n. The results resemble ideas presented by Bentley *et al.* (1978) and Müller-Hannemann and Weihe (2001), suggesting the applicability of the model in real multimodal network scenarios, even when the networks are dense and without having to apply network reduction techniques or heuristics.

5.2. *Performance in a real multimodal network*

The approach is now tested on a large multimodal network. In this test, largeness is in the sense of number of vertices and edges. The selected network scenario corresponds to the multimodal transportation system of France being one of the largest networks in Europe. The multimodal network was obtained from vector data information retrieved from a public GIS library (Geofabrik, 2010).

The network dataset for each transport choice was first processed in ArcGIS to make it suitable for computation. ArcGIS is a Windows platform application for the analysis and processing of vector geographic information system data. This application has a network analysis extension that permits the identification of junctions

Table 3. Characteristics of France multimodal network.

Modes	Number of junctions	Number of polylines	Polyline length		
			Maximum	Mean	Stnd. Dev.
Roadways	53,562	47,660	0.868028	0.010674	0.032452
Railways	18,671	20,083	1.280264	0.014966	0.046192
Motorway	7,720	7,432	1.221951	0.033488	0.078485
Waterways	17,113	11,635	3.238686	0.032070	0.095573

and polylines in each transport system (Burke (2002) describes this concept). In addition, ArcGIS also has a macro for the computation of the adjacency matrix for each system of junctions.

Table 3 summarises the number of junctions and polylines for each transport mode as well as some statistics of network's polylines. All edge lengths are given in decimal geographic degrees. Four transport modes comprise the France transport system: road, rail, waterways and motorways. The road system mainly consists of primary roads. The rail system is comprised of common train lines disregarding subway and tram. Waterways are the channels and rivers used as transportation links. Finally, the motorway system of France includes toll roads and is considered a different mode of transport in its own right. As an illustration, Fig. 6 depicts the France road system.

The construction of the multimodal network requires assembling the data for the four network modes together. This task is accomplished by an *ad-hoc* algorithm coded in the Java language. The code basically takes two inputs. These are the

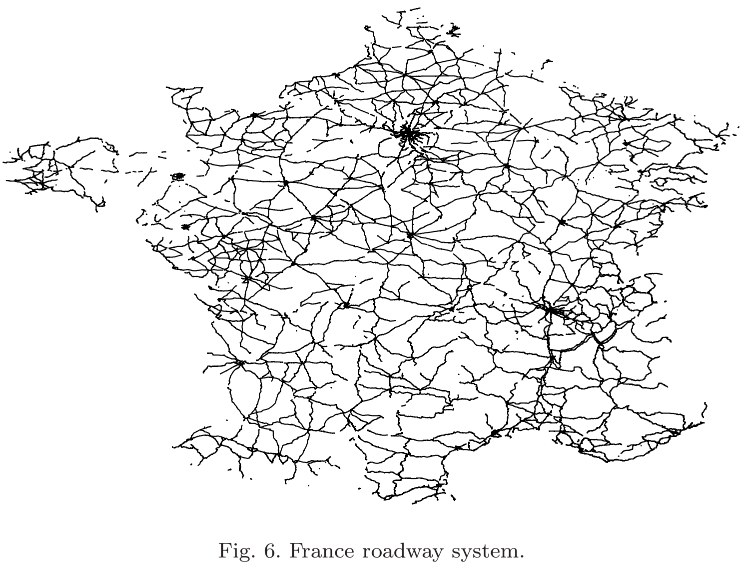

Fig. 6. France roadway system.

Table 4. Results of France multimodal network.

Cluster	Number of vertices	Number of edges	Running time	Avg. paths	Max paths
0.150	1,501	4,216	0.0170	33.0930	702
0.140	1,869	5,218	0.4993	208.270	3,320
0.130	2,343	6,280	3.1305	404.380	14,972
0.120	2,948	7,696	129.48	2013.82	37,128
0.115	3,308	8,400	557.39	3759.26	58,224

adjacency matrix of each transport mode network and a list of minimum inter-junction distances in each mode. The latter is built in ArcGIS by taking a mode junction dataset and applying the "join and relates" tool with respect to each other mode junction dataset. This information facilitates the performance of a subsequent clustering procedure used inside of the *ad-hoc* algorithm.

Two parameters need to be specified once the algorithm code is executed. A minimum clustering distance (this generates the network vertices) and a source vertex (a junction number). After entering this information, the Java code invokes the multimodal Dijkstra's algorithm, reporting at the end a list with the total number of minimal paths to each vertex together with two additional variables: The maximum number of paths found in a particular vertex and the average number of paths.

This dataset was tested by assigning a cluster distances between 0.150 and 0.115 decimal degrees (14 km to 11 km). The resulting networks together with running times (minutes) and average number of minimal paths are shown in Table 4. The computations were performed on a Xenon computer X5660 (2.8 GHz) CPU and 24 GB RAM that was set with the queue version of the multimodal Dijkstra algorithm.

Although the networks shown in Table 4 requiring longer runs of the multimodal Dijkstra's algorithm when the clustering distance is reduced, it cannot be disregarded that no constraints or reductions were required for obtaining the results in Table 4 and the cardinality of the resulting Pareto sets are quite manageable for any further analysis.

6. Conclusion

In modeling multimodal networks, current approaches for determining shortest paths rely on applying application-specific constraints or heuristics to obtain tractable problems. This paper introduces a modeling approach that keeps the multimodal traits of a network by assigning discrete color attributes to the edges, uses a partial order to obtain a Pareto set of paths of potential interest, and avoids the need for reduction techniques. Although a straightforward approach to modeling networks in which there are multiple transportation modes, it does appear to give a new perspective and truly general approach for multimodal networks. Another feature of this approach is that it results in a Pareto set, which can be further investigated without rerunning the algorithm. This opens the door to post-optimal analysis in MMN.

Despite having both the intractability of the MSPP and the impossibility of developing a polynomial time algorithm, the experimental study indicates that the cardinality of the Pareto set of minimal paths is typically low order polynomial for colored-edge graphs with random uniformly distributed weights. Furthermore, the approach can deal with networks as large as the multimodal transportation system of France without applying any reduction technique or constraint.

Future work might deal with the development of more efficient data structures for handling Pareto sets in colored-edge graphs. Furthermore, the impact of reduction techniques on the cardinality of $\mathcal{M}_{uv}$ is something worth to be investigated.

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Biography

Andrew Ensor graduated from the University of Auckland (New Zealand) with a BSc(Hons) and then completed a PhD at the University of California at Berkeley in 1995. He went on to work at Vanderbilt University in Tennessee and spent three years at the Universita di Siena in Tuscany before returning to New Zealand. In 2001, he joined AUT as a Senior Lecturer in Mathematical Sciences where he has been responsible for the undergraduate and postgraduate Computer Science major and helped develop the Applied Mathematics major, along with supervising various computer science and mathematics PhD students. His research interests include mobile and distributed systems, computer graphics, and related computing technologies which can be used for radio astronomy and SKA.

Felipe Lillo graduated from UBB University (Chile) with an Engineering degree and then completed a Master of Business at the IEDE Business School in 2002. He finished his PhD at AUT University (New Zealand) in 2011. His currently job is as lecturer at Católica del Maule University (Chile) where also works as researcher in the Faculty of Economics and Social Sciences. Research interests of Dr. Lillo include mathematical models in economics, graph applications in management and economics and simulation models for finance and auditing.