Exclusion in the teaching of the Lebesgue integral from the socio-epistemological theory

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ABSTRACT

The objective of this research was to identify, characterize and exemplify the phenomenon of exclusion caused by the School Mathematical Discourse regarding the teaching of the Lebesgue integral, based on the use of a model of exclusion from the socio-epistemological theory. In order to respond to the stated objective, a qualitative methodology was used, consisting of a documentary analysis of texts on measure theory aimed at teaching the concept of the Lebesgue integral and Lebesgue's mathematical work called *Intégrale, Longueur, Aire*. As a result of the research, the components of the Exclusion Model and the symbolic violence that this discourse exerts in the teaching of Lebesgue integral, through the imposition of certain meanings, procedures and arguments, became evident. Among the conclusions, it is expected that the results of this research will provide theoretical references to contribute to the redesign of the teaching of this integral, with the purpose of including the actors of the didactic system in the social construction of mathematical knowledge associated with the Lebesgue integral.

Keywords: Lebesgue Integral, Measure, School Mathematical Discourse, Exclusion Model, Socioepistemology.

1. INTRODUCTION

The concept of Lebesgue integral has many applications in Probability Theory, Fourier Analysis and Functional Analysis, among other domains of knowledge. Traditionally, this concept is taught in higher education in measure theory courses, where the previous study of concepts such as σ-algebra, measure of a set and measurable function, among others, are necessary to be able to perform a construction of this integral. In relation to the above stated, this research seeks to demonstrate, from the socio-epistemological theory [1], how the School Mathematical Discourse (SMD), an educational paradigm that regulates school mathematics [1], is the expression of a dominant epistemology that generates a Symbolic Violence, in the sense of Bourdieu and Passeron [2], regarding the teaching of Lebesgue integral. That is to say, it is the SMD itself that imposes meanings, procedures and arguments -understood in the sense of Del Valle [3], Mendoza and Cordero [4], Mendoza, Cordero, Solís and Gómez [5] and Cordero, Del Valle and Morales [6]- in the teaching of this concept, situation that the actors of the didactic system recognize in a hegemonic way. In other words, this research seeks to demonstrate *the phenomenon of Exclusion* -in the sense of Soto [7] and Soto and Cantoral [8]- with respect to the teaching of the Lebesgue integral concept.

There are different researches that confront the problem of the phenomenon of exclusion from a socio-epistemological perspective, and that can serve as a reference to support the idea of Exclusion presented in this research. Among these researches, there is the one conducted by Morales and Cordero [9], who study this phenomenon regarding the teaching of the concept of derivative and the one conducted by Moreno and Cantoral [10], who study the exclusion caused by the SMD in students with visual impairment, among others.

Hence, to show that there are meanings, procedures and arguments associated with the Lebesgue integral that are excluded by the SMD and that can serve the educational community as a reference for the design of teaching situations in, for example, measure theory courses.

In order to demonstrate the above objective, we will make use of the Exclusion Model proposed in [7, 8] (see Figure 1). This model shows how the consideration of the SMD, as a Reason System (RS) that generates maps that demarcate the way in which the actors of the didactic system must act, reason, give meanings and argue, produces a Symbolic Violence (SV). This situation is produced due to the imposition of certain meanings, procedures and arguments, passing over others that may be present in different types of situations, typical of diverse specific contexts, where mathematical knowledge is constructed. In this model, RS is expressed in a map composed of the characteristics of SMD, detailed in [11]: hegemonic, utilitarian, finished and continuous, lack of frames of reference and atomization in mathematical concepts and procedures.

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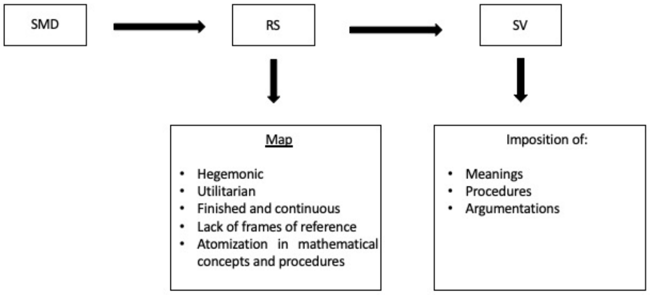

Figure 1. Exclusion Model [7, 8].

The structure of the article will be as follows: Section 2 will present the methodology that will make it possible to respond to the stated research objective. In section 3, a documentary analysis of the SMD of textbooks focused on the teaching of Lebesgue integral concept and the mathematical Lebesgue's work [12] will be presented. This will allow, as a research result, to identify, characterize and exemplify the phenomenon of Exclusion caused by the SMD in the case of teaching this concept, from the use of the Exclusion Model proposed in [7, 8]. Finally, in section 4, the conclusions of this research will be presented.

2. METHODOLOGY

Soto [7] considers the SMD of the textbooks as the invariant meanings, procedures and arguments of these texts, that is to say, those that are repeated in the different types of texts. Thus, in order to respond to the stated research objective, it is going to be necessary to infer the meanings, procedures and arguments that are invariant in the SMD of textbooks focused on the teaching of Lebesgue integral. Subsequently, the meanings, procedures and arguments present in a different scenario from the SMD of the textbooks, which, in this case, will correspond to Lebesgue's work [12] will be inferred. The confrontation of each of these analyses will make it possible to demonstrate the components of the Exclusion Model and, in this way, the Symbolic Violence that the SMD exerts in the teaching of the Lebesgue integral. A summary of the methodological scheme of this research can be seen in Figure 2.

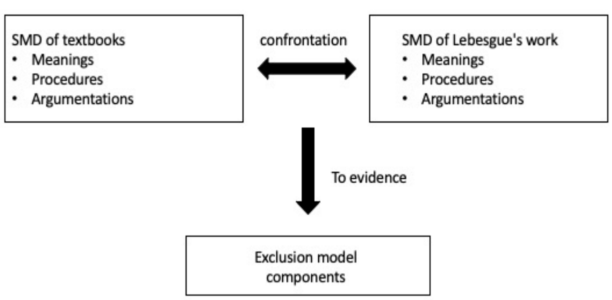

Figure 2. Methodological scheme of this research.

Data Collection

The data were collected from two sources: the first, textbooks focused on the teaching of the Lebesgue integral and the second, from the mathematical work of Henri Lebesgue, entitled *Intégrale, Longueur, Aire* [12], as it is the historical-epistemological genesis of Lebesgue Integral concept. In the case of Lebesgue's work [12], for a better understanding of the data provided by said work, and thus achieve greater precision in the analysis process, we resorted to the reading of works contemporary to [12], to achieve a greater internalization, both of its historical context, as well as its motivations and purposes. The works chosen for this purpose were those of Cauchy [13], Dirichlet [14], Riemann [15], Jordan [16] and Borel [17]. In the case of the textbooks, data were collected from three texts: Gordon [18], Mira [19] and Rebolledo [20].

Data analysis

For the analysis of the collected data, a documentary analysis technique was used [21], since it is a process that allows the information obtained from the collected data to be studied, interpreted and synthesized in detail, thus allowing better inferences to be drawn about the information underlying the documents studied. The documentary analysis was carried out separately by two groups, each made up of specialists in Mathematics and Didactics of Mathematics, who were part of the technical staff of this research. In this sense, each group carried out, in parallel, work sessions where they analyzed the data obtained from the mathematical works and study textbooks involved, in order to subsequently propose elements of the Exclusion model of Figure 1, in the case of the teaching of the Lebesgue integral. Once each group had carried out its analysis, the results of each group were discussed together, in order to recognize similarities and discrepancies in the process of documentary analysis, and thus better identify, characterize and exemplify the phenomenon of exclusion caused by the SMD in the case of the teaching of the Lebesgue integral. In other words, for the analysis of the data obtained, the data triangulation method was considered [22], which allowed minimizing the bias of a single analysis, increasing the quality and validity of the information found [23].

3. ANALYSIS AND RESULTS

Lebesgue integral in the SMD

In the SMD of the texts studied [18, 19, 20] it was observed, invariably, that the argumentation for carrying out the construction of the Lebesgue integral is to reformulate and generalize, from measure theory, the construction of the Riemann integral for a broader class of functions (called measurable functions). These textbooks invariably present the same structure to build the concept of Lebesgue integral: they start with the definition and study of the concept of σ-algebra, and then define and study the concept of measure of a set (μ), measurable function and simple measurable function. Next, we proceed to define the integral for positive, simple and measurable functions s and, after that, it is made use of the above information to define the integral of positive measurable functions f as:

$$\int_{\Omega} f d\mu = sup\left\{\int_{\Omega} s d\mu : s \text{ is simple, measurable, } 0 \leq s \leq f\right\} \quad \text{Eq. (1)}$$

Subsequently, for the case of measurable functions f of any sign, decompose this function as:

$$f = f^{+} - f^{-} \quad \text{Eq. (2)}$$

where

$$f^{+} = máx\{f, 0\}$$

$$f^{-} = -mín\{f, 0\} \quad \text{Eq. (3)}$$

and the Lebesgue integral of f is defined as:

$$\int_{\Omega} f d\mu = \int_{\Omega} f^{+} d\mu - \int_{\Omega} f^{-} d\mu$$

Eq. (4)

From the above, it can be noted that in these textbooks the procedure to construct the Lebesgue integral of a measurable function f is mainly based on the use of integrals of positive, simple and measurable functions s to mean the Lebesgue integral of f as it appears in Eq. (1), in the case of a measurable and positive function f, and as it appears in Eq. (4), in the case of a measurable function f and of any sign.

Lebesgue Integral in *Intégrale, Longeur, Aire*

For Canela [24], during the 19th century there was a process of arithmetization of mathematical analysis (and in particular, of the concept of integral), which sought to leave behind the geometric intuition, characteristic of the 18th century, which, although effective, paid little attention to the foundation and structure of the analysis itself. Cauchy [13], detaching himself from the geometry and giving an analytical treatment, constructed the concept of integral for continuous functions, and subsequently extended its definition for functions with a finite number of discontinuities. Dirichlet [14], extended Cauchy's work to functions with an infinite number of discontinuities. From this point, a question arose about the conditions that the set of discontinuity points of a function must fulfill for it to be integrable. In this direction, Riemann [15] succeeded in extending the definition of integral to functions that were even discontinuous in a dense set of points, in addition to establishing criteria for a function to be integrable. Jordan [16], through the study of the concept of content of a set, was able to study in a better way the set of discontinuities in the definite integral of a function, introducing, for the first time, a notion of measure of a set in the theory of integration, called *content of a set*.

Regarding the notion of measure, for Borel [17] the idea of measure rested on the generalization of the length of an interval. That is why, with the intention of extending this idea, he introduced the notion of Borelian or B-measurable sets and presented an axiomatic theory of the measure of a set, which had to fulfill the following properties: 1) the measure of the union of a numerable infinity of disjoint sets is equal to the sum of their measures, 2) the measure of the difference of two sets is equal to the difference of their measures and 3) the measure of a set is never negative.

What was done by Jordan [14] and Borel [17] served as a basis for the construction of the concept of integral in Lebesgue's mathematical work. In the first chapter of this mathematical work, Lebesgue [10] defined, as it was done in Borel [17], the measure of a set by means of its essential properties. Later, he indicated the relations existing between the measure thus defined and the content of a set [16]. According to Recalde [25], Lebesgue not only transcribed the developments of Jordan and Borel, but also gave them a more concrete form with the aim of providing a conceptual solution to the problem of measure in general. In the first chapter of this work, Lebesgue [12] defined, as did Borel [17], the measure of a set by means of its essential properties. Subsequently, he indicated the existing relations between the measure thus defined and Jordan's content:

> Dans le premier chapitre je définis, avec Mr. Borel, la measure d` un ensemble par, ses propriétés essentielles. Après avoir complété et précisé les indications un peu rapides que donne Mr. Borel, j´indique quelles relations il y a, entre la measure ainsi définie et la measure au sens de Mr. Jordan. [12, p. 232].

For Lebesgue [12], the basis for the construction of his integral rested on a geometric perspective (area under the curve of a continuous and bounded function), taking up this perspective that had been neglected during the 19th century. The advantage was that his method of constructing the definite integral could be extended to a wider class of functions, called *summable functions*. Although the construction of this integral was geometrically based, Lebesgue managed to give this definition an analytical character, in accordance with the rigor required at that time [24], where he used sums similar to those of Riemann to define his integral.

In summary, Lebesgue [12] sought to define a measured function m, taking values in the interval $[0, \infty]$, with the following characteristics:

L_1: $m(E) \neq 0$, for some E.
L_2: Two equal sets have the same measure.
L_3: The measure of the union of a countable finite or infinite number countable of disjoint sets is the sum of the measure of these sets.

In Recalde [25] we find an explanation of the meaning of each of these properties. For Lebesgue [12], these properties summarize the very activity of measuring. The first property avoids working with the null function, which would not be in accordance with the traditional activity of measuring. The second property is related to the invariance of the measure under translations. It should be noted that for Lebesgue the idea of two sets being equal is related to the fact that one of these sets can be moved to make it coincide with the other. Finally, the third property reflects a natural fact in the way of measuring: the measure of a whole is equal to the sum of the measures of its parts.

In the first instance, Lebesgue defined the measure of a subset of the Cartesian plane $\mathbb{R}^2$. If E is a bounded subset of $\mathbb{R}^2$, defined the outer measure of E as follows:

$$m_e(E) = \text{ínf}(B)$$

Eq. (5)

where

$$B = \{\sum m(\Delta_i) : E \subseteq \cup \Delta_i\}$$

Eq. (6)

Δ_i are triangles of the plane $\mathbb{R}^2$ and $m(\Delta_i)$ is the measure of a triangle, which coincides with its area. In addition, Lebesgue defined the interior measure of E, as:

$$m_i(E) = m(ABC) - m_e(E_{ABC}^c)$$

Eq. (7)

where ABC is a triangle containing E and $m_e(E_{ABC}^c)$ corresponds to the outer measure of the complement of E with respect to the triangle ABC. Thus, Lebesgue pointed out that E is measurable if and only if $m_e(E) = m_i(E)$, this common value is referred to as the measure of E, $m(E)$.

Lebesgue approached the problem of the integral of a function f of any sign from a geometrical perspective. For this purpose, Lebesgue considered the following subset E of the plane $\mathbb{R}^2$:

$$E = \{(x, y) \in \mathbb{R}^2 : 0 \leq yf(x), 0 \leq y^2 \leq f(x)^2, a \leq x \leq b\}$$
$$= E_1 \cup E_2 \qquad \text{Eq. (8)}$$

where

$$E_1 = \{(x, y) \in E : y \geq 0\}$$
$$E_2 = \{(x, y) \in E : y < 0\} \qquad \text{Eq. (9)}$$

Lebesgue established that a function f with any sign is Lebesgue integrable if and only if E is measurable (and therefore E_1 and E_2) and established that

$$\int_a^b f(x)dx = m(E_1) - m(E_2) \qquad \text{Eq. (10)}$$

Based on this, Lebesgue defined the summable functions as all those that make the set E measurable (and therefore E_1 and E_2) (see Figure 3)

17. Ces résultats suggèrent immédiatement la généralisation suivante: ***si l'ensemble E est mesurable, (auquel cas E_1 et E_2 le sont) nous appellerons intégrale définie de f, prise entre a et b, la quantité***

$$m(E_1) - m(E_2).$$

Les fonctions f correspondantes seront dites sommables.

Figure 3: Definition of summable function [12, p. 250].

From the previous analysis, it can be inferred that in Lebesgue's mathematical work, the argumentation to construct the concept of integral came from a geometrical perspective. The procedure was given by a classification of the set E in the subsets E_1 and E_2 given in Eq. (9) and the meaning of the integral of a summable function f, of any sign, corresponds to subtraction $m(E_1) - m(E_2)$ (see Eq. (10)).

Now, after the analysis of the texts used for the teaching of the Lebesgue integral and the historical-epistemological development of this integral [12], it became evident that the SMD argumentation of the textbooks is not only not the unique, but also imposes certain meanings, procedures and argumentations, thus generating a symbolic violence that will be detailed below, according to the Exclusion Model (see Figure 1):

Hegemonic character of the SMD. As it has been seen in the SMD of the texts studied, the argumentation that is imposed is framed in measure theory, where the aim is to reformulate and generalize the construction of the Riemann integral for a broader class of functions (called measurable functions), from which meanings and procedures are derived that are also framed within this theory. Furthermore, it is important to note that the concept σ-algebra presented in these texts did not exist at the time when Lebesgue [12] constructed his integral, due to his uncertainty regarding the existence of non-measurable sets. Moreover, Recalde [25] points out that the texts focused on the teaching of the Lebesgue integral usually hide the historical process of the construction of this integral. In this aspect, it is not explained that one of the reasons why the measured function is defined on such a fine structure as a σ -algebra, is due to the fact that it is impossible to define a measure, invariant under translations, on all the subsets of the plane $\mathbb{R}^2$.

In the analysis of Lebesgue's work [12], another type of argumentation to carry out the construction of this integral was evidenced, which came from a geometrical approach, where it was sought to build the integral from the consideration of the subset E of $\mathbb{R}^2$ indicated in Eq. (8). The foregoing allowed the generation of meanings and procedures different from those found in the SMD of the textbooks, which also come from a geometrical perspective.

Utilitarian character of the SMD. In this case, the SMD of the textbooks presents the Lebesgue integral as a generalization, framed in the measure theory, of the Riemann integral, which is a useful tool to solve certain types of problems in the field of Mathematical Analysis that the Riemann integral does not solve. In this way, the SMD does not allow students to problematize, infer and disrupt this knowledge. Thus, this knowledge does not take on a functional character for the students, understanding by functionality of mathematical knowledge a knowledge organically incorporated into the human being, which transforms their reality, as opposed to utilitarian knowledge [26]. For example, during the teaching of this concept one could problematize how to measure subsets of the Cartesian plane $\mathbb{R}^2$ that are determined by certain types of functions, such as, for example, the subset E given in Eq. (8). This would allow the emergence, on the part of the students, of meanings, procedures and arguments associated with the Lebesgue integral that are different from those imposed by the SMD.

Finished and continuous character of the SMD. This character was evidenced in the SMD by imposing, previously, the study of mathematical objects pre-existing to the action of the human being, such as σ-algebra, measurable functions and simple functions, among others, as a necessity to carry out the construction of the Lebesgue integral. In this situation, students do not have the possibility to problematize and understand the historical-epistemological genesis of these concepts, leaving them no other option but to assume the mathematical objects presented to them during the teaching processes.

Lack of frames of references in the SMD. The lack of frames of reference that would allow resignifying, in the sense of Cordero [27], the Lebesgue integral was evidenced in the lack of arguments in the SMD to construct this integral. There is no evidence in this discourse of the presence of geometrical arguments that allow students to carry out a construction of the Lebesgue integral.

Atomization of the SMD into mathematical concepts and procedures. In this case, this atomization occurs as a result of the transposition of knowledge, where it is depersonalized and decontextualized [28], reducing it to the study of mathematical objects, such as σ-algebras, measurable functions and simple functions, among others. In the case of the SMD, it was evidenced that the teaching of Lebesgue integral does not incorporate its historical-epistemological context as a reference for the teaching of this concept.

4. CONCLUSION

The SMD often neglects the contexts, communities and specific situations that make mathematical knowledge emerge, which has been reflected in the consideration of a dominant epistemology that imposes meanings, procedures and arguments to mathematical knowledge [11]. Based on the use of the Exclusion Model, this research evidenced that the SMD present in the textbooks analyzed generates a Symbolic Violence regarding the teaching of the concept of Lebesgue integral. This situation occurs in the sense that the SMD imposes meanings, procedures and arguments framed in measure theory, leaving aside the historical-epistemological genesis of this concept, in this case, the Lebesgue´s work [12], which can serve as a basis for the design of school situations that allow confronting the phenomenon of Exclusion. In this direction, it is expected that this research will provide theoretical references to contribute to the redesign of the SMD with the purpose of including the actors of the didactic system in the social construction of mathematical knowledge, in the sense of [11], associated with Lebesgue integral.

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